Saint Venant compatibility equations on a surface - Application to linear intrinsic shell theory

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SAINT VENANT EQUATIONS ON A SURFACE

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Abstract. We establish that the linearized change of metric and linearized change of curvature tensors associated with a displacement field of a surface S immersed in $\mathbb{R}^3$ must satisfy compatibility conditions that may be viewed as the linear version of the Gauss and Codazzi-Mainardi equations. These compatibility conditions, which are the analogous in two-dimensional shell theory of the Saint Venant equations in three-dimensional elasticity, constitute the Saint Venant equations on the surface S.

We next show that these compatibility conditions are also sufficient, i.e., they in fact characterize the linearized change of metric and the linearized change of curvature tensors in the following sense: If two symmetric matrix fields of order two defined over a simply-connected surface $S \subset \mathbb{R}^3$ satisfy the above compatibility conditions, then they are the linearized change of metric and linearized change of curvature tensors associated with a displacement field of the surface S, a field whose existence is thus established.

The proof provides an explicit algorithm for recovering such a displacement field from the linearized change of metric and linearized change of curvature tensors. This algorithm may be viewed as the linear counterpart of the reconstruction of a surface from its first two fundamental forms.

Résumé. On établit que le tenseur linéarisé de changement de métrique et le tenseur linéarisé de changement de courbure associés à un champ de déplacements d'une surface S immergée dans $\mathbb{R}^3$ doivent satisfaire des conditions de compatibilité qui peuvent être vues comme une version linéarisée des équations de Gauss et de Codazzi-Mainardi. Ces conditions de compatibilité, qui sont l'analogue dans la théorie bidimensionnelle de coques des équations de Saint Venant de la théorie tridimensionnelle de l'élasticité, constituent les équations de Saint Venant sur la surface S.

On démontre ensuite que ces conditions de compatibilité sont aussi suffisantes, i.e., elles caractérisent en fait les tenseurs linéarisés de changement de métrique et de courbure, dans le sens suivant: Si deux champs de matrices symétriques d'ordre deux satisfont les conditions de compatibilité ci-dessus sur une surface simplement connexe S de $\mathbb{R}^3$, alors ils sont les tenseurs linéarisés de changement de métrique et de courbure associés à un champ de déplacements de la surface S, champ dont l'existence est ainsi établie.

La preuve fournit un algorithme explicit pour la reconstruction d'un tel champ de déplacements à partir de ses tenseurs linéarisés de changement de métrique et de courbure. Cet algorithme peut être vu comme une version linéarisée de la reconstruction d'une surface à partir de ses deux premières formes fondamentales.

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1. Introduction

All the notations used, but not defined, in this introduction are defined in the next sections.

It is a classical result in differential geometry that a surface can be reconstructed, at least locally, but also globally under an additional assumption of simple-connectedness, from its first two fundamental forms, provided they satisfy ad hoc compatibility relations. More specifically, if $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are two symmetric matrix fields of order two defined over a simply-connected domain $\omega \subset \mathbb{R}^2$, the first one being of class $\mathcal{C}^2$ and positive definite at each point of ω and the second one being of class $\mathcal{C}^1$, and these matrix fields satisfy together the Gauss and Codazzi-Mainardi equations, then there exists an immersion $\boldsymbol{\theta} : \omega \to \mathbb{R}^3$ of class $\mathcal{C}^3$ such that

$$\partial_\alpha \boldsymbol{\theta} \cdot \partial_\beta \boldsymbol{\theta} = a_{\alpha\beta} \text{ and } \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \frac{\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}}{|\partial_1 \boldsymbol{\theta} \wedge \partial_2 \boldsymbol{\theta}|} = b_{\alpha\beta} \text{ in } \omega.$$

These equations mean that the first two fundamental forms of the surface $S = \boldsymbol{\theta}(\omega)$ are indeed the matrix fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$.

We recall that the Gauss and Codazzi-Mainardi equations are respectively given by

$$R^{\nu}_{\cdot\alpha\sigma\tau} = b_{\alpha\tau} b^{\nu}_{\sigma} - b_{\alpha\sigma} b^{\nu}_{\tau}, \tag{1}$$

and

$$\partial_\sigma b_{\alpha\tau} - \partial_\tau b_{\alpha\sigma} + \Gamma^{\mu}_{\alpha\tau} b_{\mu\sigma} - \Gamma^{\mu}_{\alpha\sigma} b_{\mu\tau} = 0, \tag{2}$$

where $R^{\nu}_{\cdot\alpha\sigma\tau}$ are the mixed components of the Riemann curvature tensor associated with the metric $(a_{\alpha\beta})$, defined by

$$R^{\nu}_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^{\nu}_{\alpha\tau} - \partial_\tau \Gamma^{\nu}_{\alpha\sigma} + \Gamma^{\varphi}_{\alpha\tau} \Gamma^{\nu}_{\varphi\sigma} - \Gamma^{\varphi}_{\alpha\sigma} \Gamma^{\nu}_{\varphi\tau},$$

and where $\Gamma^{\tau}_{\alpha\beta}$ are the Christoffel symbols associated with $(a_{\alpha\beta})$, defined by

$$\Gamma^{\tau}_{\alpha\beta} := \frac{1}{2} a^{\tau\sigma} (\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta}),$$

$(a^{\tau\sigma}(y))$ being the inverse of the matrix $(a_{\alpha\beta}(y))$ at each point $y \in \omega$.

The recovery of the above immersion $\boldsymbol{\theta}$ from the two fundamental forms $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ is obtained by solving first the system

$$\begin{aligned} \partial_\alpha \mathbf{a}_\beta &= \Gamma^{\sigma}_{\alpha\beta} \mathbf{a}_\sigma + b_{\alpha\beta} \mathbf{a}_3, \\ \partial_\alpha \mathbf{a}_3 &= -b^{\sigma}_{\alpha} \mathbf{a}_\sigma, \end{aligned} \tag{3}$$

where the unknowns are the three vector fields $\mathbf{a}_i \in \mathcal{C}^2(\omega; \mathbb{R}^3)$, then by solving the system

$$\partial_\alpha \boldsymbol{\theta} = \mathbf{a}_\alpha \text{ in } \omega. \tag{4}$$

The mapping $\boldsymbol{\theta} \in \mathcal{C}^3(\omega; \mathbb{R}^3)$ found in this fashion is the sought immersion. Note that the system (3) has solutions because the Gauss and Codazzi-Mainardi equations are satisfied and the system (4) has solutions because the Christoffel symbols satisfy $\Gamma^{\tau}_{\alpha\beta} = \Gamma^{\tau}_{\beta\alpha}$.

Our objective here is to establish an infinitesimal version of this result. More specifically, given a simply-connected domain $\omega \subset \mathbb{R}^2$, let $\boldsymbol{\theta} : \omega \to$

$\mathbb{R}^3$ be an immersion of class $\mathcal{C}^3$, and let $(\gamma_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{S}^2)$ and $(\rho_{\alpha\beta}) \in \mathcal{C}^1(\omega;\mathbb{S}^2)$ be two symmetric matrix fields (these regularity assumptions will be substantially weakened later). Then we show that, if these matrix fields satisfy together the following compatibility conditions, which we shall call the *Saint Venant equations on the surface* $\boldsymbol{\theta}(\omega)$,

$$\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu}$$
$$= b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha},$$
$$\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}),$$

then there exists a vector field $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ of class $\mathcal{C}^3$ such that the two fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are respectively the linearized change of metric and linearized change of curvature tensors associated with the field $\boldsymbol{\eta}$, in the sense that

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\beta + \mathbf{a}_\alpha\cdot\partial_\beta\boldsymbol{\eta}) \text{ in } \omega,$$
$$\rho_{\alpha\beta} = (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu\boldsymbol{\eta})\cdot\mathbf{a}_3 \text{ in } \omega.$$

The notations $\gamma_{\alpha\beta|\sigma}$ and $\gamma_{\alpha\beta|\sigma\tau}$ denote respectively the first and the second covariant derivative of the field $(\gamma_{\alpha\beta})$ (see Section 3), and b^{τ}_{σ} denote the mixed components of the second fundamental form of the surface $\boldsymbol{\theta}(\omega)$.

The proof of this result furnishes an explicit algorithm for recovering the vector field $\boldsymbol{\eta}$ from the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$: one first solves the system

$$\lambda_{\alpha\beta|\sigma} + b_{\alpha\sigma}\lambda_\beta - b_{\beta\sigma}\lambda_\alpha = \gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta},$$
$$\lambda_{\alpha|\sigma} + b^{\nu}_{\sigma}\lambda_{\alpha\nu} = \rho_{\sigma\alpha} - b^{\nu}_{\sigma}\gamma_{\alpha\nu},$$

where the unknowns are the antisymmetric matrix field $(\lambda_{\alpha\beta}) \in \mathcal{C}^2(\omega;\mathbb{A}^2)$ and the vector field $(\lambda_\alpha) \in \mathcal{C}^2(\omega;\mathbb{R}^2)$; then one solves the system

$$\partial_\alpha\boldsymbol{\eta} = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^\beta + \lambda_\alpha\mathbf{a}^3 \text{ in } \omega.$$

The vector field $\boldsymbol{\eta} \in \mathcal{C}^3(\omega;\mathbb{R}^3)$ found in this fashion has the desired properties.

Note that the first system has solutions because $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ satisfy the above Saint Venant equations on a surface and that the second system has solutions because the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are symmetric.

The results obtained in this article may be viewed as infinitesimal versions of the reconstruction of a surface from its fundamental forms, because the Saint Venant equations on a surface are the linear part with respect to ε of the Gauss and Codazzi-Mainardi equations associated with the immersion $(\boldsymbol{\theta} + \varepsilon\boldsymbol{\eta})$.

Note that in an earlier article, Ciarlet and Gratie [3] already found necessary and sufficient conditions for matrix fields to be linearized change of metric and change of curvature tensors on a surface, but these remained in an "abstract" form. Their approach consisted in finding these conditions as a consequence of the Saint Venant equations on a three-dimensional open set. In essence, the necessary and sufficient conditions found in this way are the Saint Venant equations associated with the Cartesian coordinates of the matrix field defined in curvilinear coordinates by

$$g_{\alpha\beta} = \gamma_{\alpha\beta} - x_3\rho_{\alpha\beta} + \frac{x_3^2}{2}(b_\alpha^\sigma \rho_{\beta\sigma} + b_\beta^\tau \rho_{\alpha\tau} - 2b_\alpha^\sigma b_\beta^\tau \gamma_{\sigma\tau}),$$
$$g_{\alpha 3} = g_{3\beta} = g_{33} = 0,$$

over the three dimesional set $\{\boldsymbol{\theta}(y) + x_3\mathbf{a}_3(y);\ y \in \omega,\ x_3 \in (-\varepsilon, \varepsilon)\}$, where $\varepsilon > 0$ is a small enough real number.

2. Notations and other preliminaries

Latin indices and exponents vary in the set $\{1, 2, 3\}$, Greek indices and exponents vary in the set $\{1, 2\}$, and the summation convention with respect to repeated indices and exponents is systematically used in conjunction with this rule.

All spaces, matrices, etc., are real. The Kronecker symbols are denoted δ_α^β or δ_i^j. The symbols $\mathbb{M}^n$, $\mathbb{A}^n$, $\mathbb{S}^n$, and $\mathbb{S}^n_>$ respectively designate the sets of all square matrices, of all antisymmetric matrices, of all symmetric matrices, and of all positive-definite symmetric matrices, of order n.

The Euclidean inner product of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$ and the Euclidean norm of $\boldsymbol{u} \in \mathbb{R}^3$ are respectively denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$ and $|\boldsymbol{u}|$. The notation $(t_{\alpha\beta})$ designates the matrix in $\mathbb{M}^2$ with $t_{\alpha\beta}$ as its elements, the first index α being the row index. The spectral norm of a matrix $\mathbf{A} \in \mathbb{M}^n$ is given by

$$|\mathbf{A}| := \sup\{|\mathbf{A}\boldsymbol{v}|;\ \boldsymbol{v} \in \mathbb{R}^n,\ |\boldsymbol{v}| \leq 1\}.$$

Let ω be an open subset of $\mathbb{R}^2$. The coordinates of a point $y \in \omega$ are denoted y_α. Partial derivative operators of order $m \geq 1$ are denoted

$$\partial^{\mathbf{k}} := \frac{\partial^{|\mathbf{k}|}}{\partial y_1^{k_1} \partial y_2^{k_2}}$$

where $\mathbf{k} = (k_\alpha) \in \mathbb{N}^2$ is a multi-index satisfying $|\mathbf{k}| := k_1 + k_2 = m$. Partial derivative operators of the first, second, and third order are also denoted $\partial_\alpha := \partial/\partial x_\alpha$, $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$, and $\partial_{\alpha\beta\tau} := \partial^3/\partial y_\alpha \partial y_\beta \partial y_\tau$.

The space of all continuous functions from a subset $X \subset \mathbb{R}^3$ into a normed space Y is denoted $\mathcal{C}^0(X; Y)$, or simply $\mathcal{C}^0(X)$ if $Y = \mathbb{R}$. For any integer $m \geq 1$, the space of all real-valued functions that are m times continuously differentiable over ω is denoted $\mathcal{C}^m(\omega)$.

The space $\mathcal{C}^m(\overline{\omega})$, $m \geq 1$, is defined as that consisting of all functions $f \in \mathcal{C}^1(\omega)$ that, together with their partial derivatives of order $\leq m$, possess continuous extentions to the closure $\overline{\omega}$ of ω. If ω is bounded, then the space $\mathcal{C}^m(\overline{\omega})$ equipped with the norm

$$\|f\|_{\mathcal{C}^m(\overline{\omega})} := \max_{|\alpha| \leq m} \Big(\sup_{y \in \omega} |\partial^\alpha f(x)| \Big)$$

is a Banach space. If Y is a normed vector space, the spaces $\mathcal{C}^m(\omega; Y)$ and $\mathcal{C}^m(\overline{\omega}; Y)$ are similarly defined.

The Lebesgue and Sobolev spaces $L^p(\omega;Y)$ and $W^{m,p}(\omega;Y)$, where $m \geq 1$ is an integer, $p \geq 1$, and Y is a normed vector space, are respectively equipped with the norms

$$\|f\|_{L^p(\omega;Y)} := \Big\{ \int_\omega |f(x)|^p dx \Big\}^{1/p},$$

and

$$\|f\|_{W^{m,p}(\omega;Y)} := \Big\{ \int_\omega \big(|f(x)|^p + \sum_{|\mathbf{k}|\leq m} |\partial^{\mathbf{k}} f(x)|^p\big) dx \Big\}^{1/p}.$$

We also let $L^p(\omega) := L^p(\omega;\mathbb{R})$, $W^{m,p}(\omega) := W^{m,p}(\omega;\mathbb{R})$, and $H^m(\omega;Y) = W^{m,2}(\omega;Y)$.

The space $W^{m,p}_{\rm loc}(\omega;Y)$ is the space of all mesurable functions such that $f \in W^{m,p}(U;Y)$ for all bounded open subsets $U \subset \mathbb{R}^2$ that satisfy $\overline{U} \subset \omega$.

The space of all indefinitely derivable functions $\varphi : \omega \to \mathbb{R}$ with compact support contained in ω is denoted $\mathcal{D}(\omega)$ and the space of all distributions over ω is denoted $\mathcal{D}'(\omega)$. The closure of $\mathcal{D}(\omega)$ in $H^m(\omega)$ is denoted $H^m_0(\omega)$. Similar definitions hold for the spaces $H^m_0(\omega;\mathbb{R}^m)$, $H^m_0(\omega;\mathbb{S}^m)$, etc. The dual of the space $H^m_0(\omega)$ is denoted $H^{-m}(\omega)$.

The following technical result will be needed later.

Lemma 1. *Let Ω be an open subset of $\mathbb{R}^2$.*

a) If $f \in \mathcal{C}^1(\overline{\omega})$ and $\chi \in H^{-1}(\omega)$, then the mapping

$$\varphi \in H^1_0(\omega) \mapsto < \chi, f\varphi > \in \mathbb{R}$$

belongs to $H^{-1}(\omega)$ and is denoted $f\chi$.

b) If $f \in \mathcal{C}^2(\overline{\omega})$ and $\chi \in H^{-2}(\omega)$, then the mapping

$$\varphi \in H^2_0(\omega) \mapsto < \chi, f\varphi > \in \mathbb{R}$$

belongs to $H^{-2}(\omega)$ and is denoted $f\chi$.

Proof. We only need to prove the continuity of the mappings defined in the lemma. If $f \in \mathcal{C}^1(\overline{\omega})$ and $\chi \in H^{-1}(\omega)$, then there exists a constant C_1 such that

$$| < \chi, f\varphi > | \leq \|\chi\|_{H^{-1}(\omega)} \|f\varphi\|_{H^1(\omega)} \leq C_1 \|\chi\|_{H^{-1}(\omega)} \|f\|_{\mathcal{C}^1(\overline{\omega})} \|\varphi\|_{H^1(\omega)}$$

for all $\varphi \in H^1_0(\omega)$. This means that $f\chi \in H^{-1}(\omega)$.

Likewise, if $f \in \mathcal{C}^2(\overline{\omega})$ and $\chi \in H^{-2}(\omega)$, then there exists a constant C_2 such that

$$| < \chi, f\varphi > | \leq \|\chi\|_{H^{-2}(\omega)} \|f\varphi\|_{H^2(\omega)} \leq C_2 \|\chi\|_{H^{-2}(\omega)} \|f\|_{\mathcal{C}^2(\overline{\omega})} \|\varphi\|_{H^2(\omega)}$$

for all $\varphi \in H^2_0(\omega)$. This means that $f\chi \in H^{-2}(\omega)$.

$\square$

Remark. In other words, this lemma asserts that if $f \in \mathcal{C}^1(\overline{\omega})$ and $\chi \in H^{-1}(\omega)$, then the product $f\chi$ is well defined as an element of $H^{-1}(\omega)$; and likewise, if $f \in \mathcal{C}^2(\overline{\omega})$ and $\chi \in H^{-2}(\omega)$, then the product $f\chi$ is well defined as an element of $H^{-2}(\omega)$. $\square$

A *domain* in $\mathbb{R}^2$ is a bounded and connected open set with a Lipschitz-continuous boundary, the set ω being locally on the same side of its boundary. The definition of such a boundary is the usual one, as found for instance in Adams [1], Grisvard [5], or Nečas [6].

We conclude this section with Poincaré Theorem, which is valid classically only for continuously differentiable functions, but was generalized in Ciarlet & Ciarlet, Jr. [2] as follows.

Theorem 1. *Let ω be a simply-connected domain of $\mathbb{R}^2$. Let $h_\alpha \in H^{-1}(\omega)$ be distributions that satisfy*

$$\partial_\beta h_\alpha = \partial_\alpha h_\beta \text{ in } H^{-2}(\omega).$$

Then there exists a function $p \in L^2(\omega)$, unique up to an additive constant, such that

$$h_\alpha = \partial_\alpha p \text{ in } H^{-1}(\omega).$$

Obviously, this theorem remains valid if the functions h_α are replaced with matrix fields $\mathbf{H}_\alpha \in H^{-1}(\omega; \mathbb{M}^2)$, the solution p being then replaced by a matrix field $\mathbf{P} \in L^2(\omega; \mathbb{M}^2)$. If the matrix fields $\mathbf{H}_\alpha$ are anti-symmetric (resp. symmetric), then the matrix field $\mathbf{P}$ is also anti-symmetric (resp. symmetric).

3. Curvilinear coordinates on a surface

A mapping $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega}; \mathbb{R}^3)$ is an *immersion* if the vectors $\partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y \in \overline{\omega}$.

Let ω be an open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion. Then the image $S := \boldsymbol{\theta}(\omega)$ is a surface immersed in $\mathbb{R}^3$. For each $y \in \omega$, the vectors

$$\mathbf{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$$

form a basis in the tangent space to the surface $\boldsymbol{\theta}(\omega)$ at the point $\boldsymbol{\theta}(y)$. The tangent vector fields $\mathbf{a}^\beta$, defined by

$$\mathbf{a}_\alpha(y) \cdot \mathbf{a}^\beta(y) = \delta_\alpha^\beta \text{ for all } y \in \omega,$$

form the dual bases. A unit normal vector to S at $\boldsymbol{\theta}(y)$ is defined by

$$\mathbf{a}_3(y) = \mathbf{a}^3(y) := \frac{\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)}{|\mathbf{a}_1(y) \wedge \mathbf{a}_2(y)|}.$$

Note that at each point $\boldsymbol{\theta}(y)$ of S the vectors $(\mathbf{a}_1(y), \mathbf{a}_2(y), \mathbf{a}_3(y))$ form a basis in $\mathbb{R}^3$ and that $(\mathbf{a}^1(y), \mathbf{a}^2(y), \mathbf{a}^3(y))$ form its dual bases. As a consequence, any vector field $\boldsymbol{\eta} : \omega \to \mathbb{R}^3$ can be written as a linear combination of the vector fields $\mathbf{a}^i$ as

$$\boldsymbol{\eta} = (\boldsymbol{\eta} \cdot \mathbf{a}_i)\mathbf{a}^i.$$

The covariant components of the first fundamental form of S are defined by

$$a_{\alpha\beta}(y) = \mathbf{a}_\alpha(y) \cdot \mathbf{a}_\beta(y) \text{ for all } y \in \omega,$$

and the contravariant components of the same form are defined by

$$a^{\alpha\beta}(y) = \mathbf{a}^\alpha(y) \cdot \mathbf{a}^\beta(y),$$

or equivalently, by $(a^{\alpha\beta}(y) = (a_{\sigma\tau}(y))^{-1}$ in ω.

The covariant components of the second fundamental form of S are defined by

$$b_{\alpha\beta}(y) = -\partial_\alpha \mathbf{a}_3(y) \cdot \mathbf{a}_\beta(y) = \partial_\alpha \mathbf{a}_\beta(y) \cdot \mathbf{a}_3(y) \text{ for all } y \in \omega,$$

and the mixed components of the same form are defined by

$$b_\alpha^\tau = -\partial_\alpha \mathbf{a}_3(y) \cdot \mathbf{a}^\tau(y) = \partial_\alpha \mathbf{a}^\tau(y) \cdot \mathbf{a}_3(y) \text{ for all } y \in \omega,$$

or equivalently, by $b_\alpha^\tau = a^{\tau\beta} b_{\alpha\beta}$ for all $y \in \omega$.

The Christoffel symbols on the surface S are defined by

$$\Gamma^\tau_{\alpha\beta} := \frac{1}{2} a^{\tau\nu} \left(\partial_\alpha a_{\beta\nu} + \partial_\beta a_{\alpha\nu} - \partial_\nu a_{\alpha\beta} \right) \text{ in } \omega.$$

Note that the Christoffel symbols verify $\Gamma^\tau_{\alpha\beta} = \Gamma^\tau_{\beta\alpha}$. The regularity assumption on the immersion $\boldsymbol{\theta}$ implies that the functions $a_{\alpha\beta}$ and $a^{\tau\nu}$ belong to the space $\mathcal{C}^2(\overline{\omega})$, which in turn implies that $\Gamma^\tau_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$.

It is well known that the derivatives of the vector fields $\mathbf{a}_i$ satisfy the equations of Gauss and Weingarten:

$$\begin{aligned} \partial_\alpha \mathbf{a}_\beta &= \Gamma^\nu_{\alpha\beta} \mathbf{a}_\nu + b_{\alpha\beta} \mathbf{a}^3, \\ \partial_\alpha \mathbf{a}_3 &= -b_\alpha^\nu \mathbf{a}_\nu, \end{aligned}$$

from which a straightforward computation shows that the derivatives of the vector fields $\mathbf{a}^j$ satisfy

$$\begin{aligned} \partial_\alpha \mathbf{a}^\tau &= -\Gamma^\tau_{\alpha\nu} \mathbf{a}^\nu + b_\alpha^\tau \mathbf{a}^3, \\ \partial_\alpha \mathbf{a}^3 &= -b_{\alpha\nu} \mathbf{a}^\nu. \end{aligned}$$

These equations, combined with the commutativity of the second derivatives of the vector field $\mathbf{a}_\alpha$ (i.e., $\partial_\tau(\partial_\sigma \mathbf{a}_\alpha) = \partial_\sigma(\partial_\tau \mathbf{a}_\alpha)$), imply that

$$\begin{aligned} &(\partial_\tau \Gamma^\nu_{\sigma\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\nu_{\tau\mu} - b_{\sigma\alpha} b_\tau^\nu) \mathbf{a}_\nu + (\partial_\tau b_{\sigma\alpha} + \Gamma^\mu_{\sigma\alpha} b_{\tau\mu}) \mathbf{a}_3 \\ &\qquad = (\partial_\sigma \Gamma^\nu_{\tau\alpha} + \Gamma^\mu_{\tau\alpha} \Gamma^\nu_{\sigma\mu} - b_{\tau\alpha} b_\sigma^\nu) \mathbf{a}_\nu + (\partial_\sigma b_{\tau\alpha} + \Gamma^\mu_{\tau\alpha} b_{\sigma\mu}) \mathbf{a}_3. \end{aligned}$$

These relations are equivalent with the *Gauss and Codazzi-Mainardi equations*:

$$R^\nu_{\cdot\alpha\sigma\tau} = b_{\tau\alpha} b_\sigma^\nu - b_{\sigma\alpha} b_\tau^\nu, \tag{5}$$

$$\partial_\sigma b_{\tau\alpha} - \partial_\tau b_{\sigma\alpha} + \Gamma^\mu_{\tau\alpha} b_{\sigma\mu} - \Gamma^\mu_{\sigma\alpha} b_{\tau\mu} = 0, \tag{6}$$

where

$$R^\nu_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^\nu_{\tau\alpha} - \partial_\tau \Gamma^\nu_{\sigma\alpha} + \Gamma^\mu_{\tau\alpha} \Gamma^\nu_{\sigma\mu} - \Gamma^\mu_{\sigma\alpha} \Gamma^\nu_{\tau\mu}$$

are the mixed components of the Riemann curvature tensor associated with the metric $(a_{\alpha\beta})$.

The covariant derivatives of a 1-covariant tensor field $(\eta_\alpha) \in H^1(\omega;\mathbb{R}^2)$ are defined by

$$\eta_{\alpha|\beta} := \partial_\beta \eta_\alpha - \Gamma^\nu_{\beta\alpha}\eta_\nu,$$

or, equivalently, by (the first term of the right-hand side is defined by the other terms)

$$\partial_\beta(\eta_\alpha \mathbf{a}^\alpha) = \eta_{\alpha|\beta}\mathbf{a}^\alpha + b^\nu_\beta \eta_\nu \mathbf{a}^3. \tag{7}$$

The covariant derivatives of a 2-covariant tensor field $(T_{\alpha\beta}) \in L^2(\omega;\mathbb{M}^2)$ are defined by

$$T_{\alpha\beta|\sigma} := \partial_\sigma T_{\alpha\beta} - \Gamma^\nu_{\sigma\alpha}T_{\nu\beta} - \Gamma^\nu_{\sigma\beta}T_{\alpha\nu}.$$

Note that each distribution $T_{\alpha\beta|\sigma}$ is well defined in the space $H^{-1}(\omega)$. Since the matrix fields

$$\mathbf{a}^i \otimes \mathbf{a}^j := \mathbf{a}^i(\mathbf{a}^j)^T$$

form a basis in the space $\mathcal{C}^2(\overline{\omega};\mathbb{M}^3)$ and since

$$\begin{aligned}
&\partial_\sigma(\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = -\Gamma^\alpha_{\sigma\tau}\mathbf{a}^\tau \otimes \mathbf{a}^\beta - \Gamma^\beta_{\sigma\tau}\mathbf{a}^\alpha \otimes \mathbf{a}^\tau + b^\alpha_\sigma \mathbf{a}^3 \otimes \mathbf{a}^\beta - b^\beta_\sigma \mathbf{a}^\alpha \otimes \mathbf{a}^3,\\
&\partial_\sigma(\mathbf{a}^\alpha \otimes \mathbf{a}^3) = -\Gamma^\alpha_{\sigma\tau}\mathbf{a}^\tau \otimes \mathbf{a}^3 + b^\alpha_\sigma \mathbf{a}^3 \otimes \mathbf{a}^3 - b_{\sigma\tau}\mathbf{a}^\alpha \otimes \mathbf{a}^\tau,\\
&\partial_\sigma(\mathbf{a}^3 \otimes \mathbf{a}^\beta) = -b_{\sigma\tau}\mathbf{a}^\tau \otimes \mathbf{a}^\beta - \Gamma^\beta_{\sigma\tau}\mathbf{a}^3 \otimes \mathbf{a}^\tau + b^\beta_\sigma \mathbf{a}^3 \otimes \mathbf{a}^3,\\
&\partial_\sigma(\mathbf{a}^3 \otimes \mathbf{a}^3) = -b_{\sigma\tau}\mathbf{a}^\tau \otimes \mathbf{a}^3 - b_{\sigma\tau}\mathbf{a}^3 \otimes \mathbf{a}^\tau
\end{aligned}$$

the above definition of the covariant derivatives $T_{\alpha\beta|\sigma}$ is equivalent with the relations

$$\partial_\sigma(T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = T_{\alpha\beta|\sigma}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + b^\alpha_\sigma T_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^\beta + b^\beta_\sigma T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^3. \tag{8}$$

Note that these equations are to be understood in the distributional sense, the functions $T_{\alpha\beta}$ being only in $L^2(\omega)$.

Finally, for all 3-covariant tensor fields $(T_{\alpha\beta\sigma})$ with components in $H^{-1}(\omega)$, the covariant derivatives are defined by

$$T_{\alpha\beta\sigma|\tau} := \partial_\tau T_{\alpha\beta\sigma} - \Gamma^\nu_{\tau\alpha}T_{\nu\beta\sigma} - \Gamma^\nu_{\tau\beta}T_{\alpha\nu\sigma} - \Gamma^\nu_{\tau\sigma}T_{\alpha\beta\nu}.$$

In view of Lemma 1, these covariant derivatives are in $H^{-2}(\omega)$.

Note that the Codazzi-Mainardi equations are equivalently expressed in terms of the covariant derivative in the remarquably simple form

$$b_{\alpha\sigma|\tau} = b_{\alpha\tau|\sigma},$$

or equivalently, by

$$b^\alpha_\sigma|_\tau = b^\alpha_\tau|_\sigma,$$

where the covariant derivatives $b^\alpha_\sigma|_\tau$ are defined by

$$b^\alpha_\sigma|_\tau := \partial_\tau b^\alpha_\sigma - \Gamma^\mu_{\tau\sigma}b^\sigma_\mu + \Gamma^\alpha_{\tau\mu}b^\mu_\sigma.$$

The second-order covariant derivatives of a 1-covariant tensor field $(\eta_\alpha) \in H^1(\omega;\mathbb{R}^2)$ are defined by the relations

$$\eta_{\alpha|\sigma\tau} := \partial_\tau \eta_{\alpha|\sigma} - \Gamma^\nu_{\tau\alpha}\eta_{\nu|\sigma} - \Gamma^\nu_{\tau\sigma}\eta_{\alpha|\nu}.$$

Note that these relations are well defined in $H^{-1}(\omega)$. In view of relation (7), it is then seen that these second-order covariant derivatives are also uniquely determined by the relations

$$\begin{aligned}\partial_{\tau\sigma}(\eta_\alpha \mathbf{a}^\alpha) = (\eta_{\alpha|\sigma\tau} + \Gamma^\mu_{\tau\sigma}\eta_{\alpha|\mu} - b_{\alpha\tau}b^\nu_\sigma \eta_\nu)\mathbf{a}^\alpha \\ + (b^\alpha_\tau \eta_{\alpha|\sigma} + b^\alpha_\sigma \eta_{\alpha|\tau} + (b^\alpha_{\sigma|\tau} + \Gamma^\alpha_{\tau\sigma})\eta_\sigma)\mathbf{a}^3.\end{aligned} \tag{9}$$

To see this, we first infer from relation (7) that

$$\begin{aligned}\partial_{\tau\sigma}(\eta_\alpha \mathbf{a}^\alpha) = \partial_\tau(\eta_{\alpha|\sigma}\mathbf{a}^\alpha + b^\nu_\sigma \eta_\nu \mathbf{a}^3) = \partial_\tau \eta_{\alpha|\sigma}\mathbf{a}^\alpha - \Gamma^\alpha_{\tau\nu}\eta_{\alpha|\sigma}\mathbf{a}^\nu + b^\alpha_\tau \eta_{\alpha|\sigma}\mathbf{a}^3 \\ + (\partial_\tau b^\nu_\sigma \eta_\nu + b^\nu_\sigma \partial_\tau \eta_\nu)\mathbf{a}^3 - b_{\tau\mu}b^\nu_\sigma \eta_\nu \mathbf{a}^\mu,\end{aligned}$$

from which the relations (9) are easily deduced by using the definition of the covariant derivatives $\eta_{\alpha|\sigma}$ and $\eta_{\alpha|\sigma\tau}$ and of the covariant derivatives

$$b^\nu_\sigma|_\tau := \partial_\tau b^\nu_\sigma - \Gamma^\alpha_{\tau\sigma}b^\nu_\alpha + \Gamma^\nu_{\tau\alpha}b^\alpha_\sigma.$$

An important consequence of relation (9), combined with the commutativity of the second-order derivatives of $\eta_\alpha \mathbf{a}^\alpha$, is that the second-order covariant derivatives $\eta_{\alpha|\sigma\tau}$ satisfy the property

$$\eta_{\alpha|\sigma\tau} - b_{\alpha\tau}b^\nu_\sigma \eta_\nu = \eta_{\alpha|\tau\sigma} - b_{\alpha\sigma}b^\nu_\tau \eta_\nu,$$

which is equivalent with

$$\eta_{\alpha|\sigma\tau} - \eta_{\alpha|\tau\sigma} = (b_{\alpha\tau}b^\nu_\sigma - b_{\alpha\sigma}b^\nu_\tau)\eta_\nu.$$

In view of the Gauss equation (5), this equation is equivalent with

$$\eta_{\alpha|\sigma\tau} - \eta_{\alpha|\tau\sigma} = R^{\nu}_{\cdot\alpha\sigma\tau}\eta_\nu,$$

which is the *Ricci formula* applied to the 1-covariant tensor field (η_α).

Naturally, the second-order covariant derivatives of $(T_{\alpha\beta}) \in L^2(\omega;\mathbb{M}^2)$ are defined by the relations

$$T_{\alpha\beta|\sigma\tau} := \partial_\tau T_{\alpha\beta|\sigma} - \Gamma^\nu_{\tau\alpha}T_{\nu\beta|\sigma} - \Gamma^\nu_{\tau\beta}T_{\alpha\nu|\sigma} - \Gamma^\nu_{\tau\sigma}T_{\alpha\beta|\nu},$$

It is then easily seen, in view of relation (8), that these second-order covariant derivatives are uniquely determined by the relations

$$\begin{aligned}\partial_{\tau\sigma}(T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = (T_{\alpha\beta|\sigma\tau} + \Gamma^\mu_{\tau\sigma}T_{\alpha\beta|\mu} - b^\mu_\sigma(b_{\tau\alpha}T_{\mu\beta} + b_{\tau\beta}T_{\alpha\mu}))\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ + (...)\mathbf{a}^\alpha \otimes \mathbf{a}^3 + (...)\mathbf{a}^3 \otimes \mathbf{a}^\beta + (...)\mathbf{a}^3 \otimes \mathbf{a}^3.\end{aligned} \tag{10}$$

Indeed, relation (8) implies that

$$\partial_{\tau\sigma}(T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = \partial_\tau(T_{\alpha\beta|\sigma}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + b^\alpha_\sigma T_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^\beta + b^\beta_\sigma T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^3).$$

But the terms appearing in the right-hand side of this equation satisfy

$$\begin{aligned}\partial_\tau(T_{\alpha\beta|\sigma}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = (\partial_\tau T_{\alpha\beta|\sigma} - \Gamma^\mu_{\tau\alpha}T_{\mu\beta|\sigma} - \Gamma^\mu_{\tau\beta}T_{\alpha\mu|\sigma})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ + (b^\alpha_\tau T_{\alpha\beta|\sigma})\mathbf{a}^3 \otimes \mathbf{a}^\beta + (b^\beta_\tau T_{\alpha\beta|\sigma})\mathbf{a}^\alpha \otimes \mathbf{a}^3\end{aligned}$$

and

$$\begin{aligned}\partial_\tau(b^\alpha_\sigma T_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^\beta + b^\beta_\sigma T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^3) &= -b^\mu_\sigma(b_{\tau\alpha}T_{\mu\beta} + b_{\tau\beta}T_{\alpha\mu})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &+ (b^\alpha_\sigma b^\beta_\tau + b^\alpha_\tau b^\beta_\sigma)T_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^3 + (b^\beta_\sigma T_{\alpha\beta|\tau} + (b^\beta_{\sigma|\tau} + \Gamma^\mu_{\tau\sigma}b^\beta_\mu)T_{\alpha\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^3 \\ &+ (b^\alpha_\sigma T_{\alpha\beta|\tau} + (b^\alpha_{\sigma|\tau} + \Gamma^\mu_{\tau\sigma}b^\alpha_\mu)T_{\alpha\beta})\mathbf{a}^3 \otimes \mathbf{a}^\beta.\end{aligned}$$

Hence

$$\begin{aligned}\partial_{\tau\sigma}(T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) = (T_{\alpha\beta|\sigma\tau} &+ \Gamma^\mu_{\tau\sigma}T_{\alpha\beta|\mu} - b^\mu_\sigma(b_{\tau\alpha}T_{\mu\beta} + b_{\tau\beta}T_{\alpha\mu}))\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &+ (b^\beta_\tau T_{\alpha\beta|\sigma} + b^\beta_\sigma T_{\alpha\beta|\tau} + (b^\beta_{\sigma|\tau} + \Gamma^\mu_{\tau\sigma}b^\beta_\mu)T_{\alpha\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^3 \\ &+ (b^\alpha_\tau T_{\alpha\beta|\sigma} + b^\alpha_\sigma T_{\alpha\beta|\tau} + (b^\alpha_{\sigma|\tau} + \Gamma^\mu_{\tau\sigma}b^\alpha_\mu)T_{\alpha\beta})\mathbf{a}^3 \otimes \mathbf{a}^\beta \\ &+ (b^\alpha_\sigma b^\beta_\tau + b^\alpha_\tau b^\beta_\sigma)T_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^3.\end{aligned}$$

Note that the commutativity of the second-order derivatives of $T_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta$ combined with relation (10) imply that the second-order covariant derivatives of $(T_{\alpha\beta})$ satisfy

$$T_{\alpha\beta|\sigma\tau} - b^\mu_\sigma b_{\tau\alpha}T_{\mu\beta} - b^\mu_\sigma b_{\tau\beta}T_{\alpha\mu} = T_{\alpha\beta|\tau\sigma} - b^\mu_\tau b_{\sigma\alpha}T_{\mu\beta} - b^\mu_\tau b_{\sigma\beta}T_{\alpha\mu},$$

or equivalently

$$T_{\alpha\beta|\sigma\tau} - T_{\alpha\beta|\tau\sigma} = (b^\mu_\sigma b_{\tau\alpha} - b^\mu_\tau b_{\sigma\alpha})T_{\mu\beta} + (b^\mu_\sigma b_{\tau\beta} - b^\mu_\tau b_{\sigma\beta})T_{\alpha\mu}.$$

In view of the Gauss equation (5), the last equation is the same as

$$T_{\alpha\beta|\sigma\tau} - T_{\alpha\beta|\tau\sigma} = R^{\mu}_{\cdot\alpha\sigma\tau}T_{\mu\beta} + R^{\mu}_{\cdot\beta\sigma\tau}T_{\alpha\mu},$$

which is the *Ricci formula* applied to the tensor $(T_{\alpha\beta})$.

4. Saint Venant equations on a surface

Let ω be a bounded open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega};\mathbb{R}^2)$ be an immersion. The vector fields $\mathbf{a}_i \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ and $\mathbf{a}^i \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ are defined as in Section 3.

With every vector field $\boldsymbol{\eta} \in H^1(\omega;\mathbb{R}^3)$, we associate the *linearized change of metric* tensor field, defined by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\partial_\alpha\boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta\boldsymbol{\eta}),$$

and the *linearized change of curvature* tensor field, defined by

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\nu_{\alpha\beta}\partial_\nu\boldsymbol{\eta}) \cdot \mathbf{a}_3.$$

Note that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in H^{-1}(\omega)$ and that $\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\beta\alpha}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\beta\alpha}(\boldsymbol{\eta})$.

The next theorem establishes an important property of these tensors, namely that they satisfy equations (11), which constitute the *Saint Venant equations on a surface.*

Theorem 2. *The linearized change of metric tensor $\gamma_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega;\mathbb{S}^2)$ and the linearized change of curvature tensor $\rho_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in H^{-1}(\omega;\mathbb{S}^2)$ associated with a vector field $\boldsymbol{\eta} \in H^1(\omega;\mathbb{R}^3)$ satisfy*

$$\begin{aligned}\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu}&\\ = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha}&\\ \rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu})&\end{aligned} \tag{11}$$

in the distributional sense.

Proof. Given a vector field $\boldsymbol{\eta} \in H^1(\omega;\mathbb{R}^3)$, let

$$\begin{aligned}\gamma_{\alpha\beta} &:= \frac{1}{2}(\eta_{\beta|\alpha} + \eta_{\alpha|\beta}) = \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\beta + \mathbf{a}_\alpha\cdot\partial_\beta\boldsymbol{\eta}) \in L^2(\omega),\\ \lambda_{\alpha\beta} &:= \frac{1}{2}(\eta_{\beta|\alpha} - \eta_{\alpha|\beta}) = \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\beta - \mathbf{a}_\alpha\cdot\partial_\beta\boldsymbol{\eta}) \in L^2(\omega),\\ \lambda_\alpha &:= \eta_{3|\alpha} = \partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_3 \in L^2(\omega).\end{aligned}$$

Note that $(\gamma_{\alpha\beta})$ and $(\lambda_{\alpha\beta})$ are respectively the symmetric and the antisymmetric parts of the tensor $(\eta_{\alpha|\beta})$; in particular then,

$$\lambda_{11} = \lambda_{22} = 0 \text{ and } \lambda_{12} = -\lambda_{21}.$$

The derivatives in the distributional sense of the vector field $\boldsymbol{\eta}$ are then expressed in terms of the functions $\gamma_{\alpha\beta}, \lambda_{\alpha\beta}$, and λ_α by

$$\partial_\alpha\boldsymbol{\eta} = (\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_i)\mathbf{a}^i = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^\beta + \lambda_\alpha\mathbf{a}^3 \text{ in } L^2(\omega;\mathbb{R}^3).$$

This shows that the derivatives $\partial_\alpha\boldsymbol{\eta}$ are completely determined by the symmetric tensor $(\gamma_{\alpha\beta})$ and the antisymmetric tensor $(\lambda_{\alpha\beta})$ and the vector (λ_α). In fact, they are determined only by the tensors $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$, because we now show that $(\lambda_{\alpha\beta})$ and (λ_α) are related to the tensors $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$, by the equations

$$\begin{aligned}\lambda_{\alpha\beta|\sigma} + b_{\alpha\sigma}\lambda_\beta - b_{\beta\sigma}\lambda_\alpha &= \gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta},\\ \lambda_{\alpha|\sigma} + b^\nu_\sigma\lambda_{\alpha\nu} &= \rho_{\sigma\alpha} - b^\nu_\sigma\gamma_{\alpha\nu}.\end{aligned} \tag{12}$$

Note that this system in fact reduces to the following system

$$\begin{aligned}\lambda_{12|\sigma} + b_{1\sigma}\lambda_2 - b_{2\sigma}\lambda_1 &= \gamma_{\sigma2|1} - \gamma_{\sigma1|2},\\ \lambda_{\alpha|\sigma} + b^\nu_\sigma\lambda_{\alpha\nu} &= \rho_{\sigma\alpha} - b^\nu_\sigma\gamma_{\alpha\nu},\end{aligned}$$

which has only three unknowns, namely λ_1, λ_2, and λ_{12}.

Using the relation $\partial_\alpha\mathbf{a}_\beta = \partial_\beta\mathbf{a}_\alpha$, itself a consequence of the commutativity of the second-order derivatives of the field $\boldsymbol{\theta}$, we deduce from the definition of the functions $\lambda_{\alpha\beta}$ that

$$\begin{aligned}2\partial_\sigma\lambda_{\alpha\beta} &= \partial_{\sigma\alpha}\boldsymbol{\eta}\cdot\mathbf{a}_\beta + \partial_\alpha\boldsymbol{\eta}\cdot\partial_\sigma\mathbf{a}_\beta - \partial_{\sigma\beta}\boldsymbol{\eta}\cdot\mathbf{a}_\alpha - \partial_\beta\boldsymbol{\eta}\cdot\partial_\sigma\mathbf{a}_\alpha\\ &= \partial_\alpha(2\gamma_{\sigma\beta} - \partial_\beta\boldsymbol{\eta}\cdot\mathbf{a}_\sigma) - \partial_\beta(2\gamma_{\sigma\alpha} - \partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\sigma) + \partial_\alpha\boldsymbol{\eta}\cdot\partial_\sigma\mathbf{a}_\beta - \partial_\beta\boldsymbol{\eta}\cdot\partial_\sigma\mathbf{a}_\alpha\\ &= 2(\partial_\alpha\gamma_{\sigma\beta} - \partial_\beta\gamma_{\sigma\alpha} + \partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\mathbf{a}_\sigma - \partial_\beta\boldsymbol{\eta}\cdot\partial_\alpha\mathbf{a}_\sigma),\end{aligned}$$

all equalities being valid in the distributional sense. Combining this last equality with the relations

$$\begin{aligned}\partial_\alpha\boldsymbol{\eta}\cdot\partial_\beta\mathbf{a}_\sigma &= \Gamma^\nu_{\beta\sigma}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\nu)+b_{\beta\sigma}(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_3)=\Gamma^\nu_{\beta\sigma}(\gamma_{\alpha\nu}+\lambda_{\alpha\nu})+b_{\beta\sigma}\lambda_\alpha,\\ \partial_\beta\boldsymbol{\eta}\cdot\partial_\alpha\mathbf{a}_\sigma &= \Gamma^\nu_{\alpha\sigma}(\partial_\beta\boldsymbol{\eta}\cdot\mathbf{a}_\nu)+b_{\alpha\sigma}(\partial_\beta\boldsymbol{\eta}\cdot\mathbf{a}_3)=\Gamma^\nu_{\alpha\sigma}(\gamma_{\beta\nu}+\lambda_{\beta\nu})+b_{\alpha\sigma}\lambda_\beta,\end{aligned}$$

we next deduce that

$$\begin{aligned}&\partial_\sigma\lambda_{\alpha\beta}-\Gamma^\nu_{\beta\sigma}\lambda_{\alpha\nu}+\Gamma^\nu_{\alpha\sigma}\lambda_{\beta\nu}\\ &\qquad=(\partial_\alpha\gamma_{\sigma\beta}-\Gamma^\nu_{\alpha\sigma}\gamma_{\beta\nu})-(\partial_\beta\gamma_{\sigma\alpha}-\Gamma^\nu_{\beta\sigma}\gamma_{\alpha\nu})+b_{\beta\sigma}\lambda_\alpha-b_{\alpha\sigma}\lambda_\beta\\ &\qquad=\gamma_{\sigma\beta|\alpha}-\gamma_{\sigma\alpha|\beta}+b_{\beta\sigma}\lambda_\alpha-b_{\alpha\sigma}\lambda_\beta.\end{aligned}$$

But the first term is equal to the covariant derivative $\lambda_{\alpha\beta|\sigma}$, since $\lambda_{\beta\nu}=-\lambda_{\nu\beta}$. Hence the previous equality becomes

$$\lambda_{\alpha\beta|\sigma}=\gamma_{\sigma\beta|\alpha}-\gamma_{\sigma\alpha|\beta}+b_{\beta\sigma}\lambda_\alpha-b_{\alpha\sigma}\lambda_\beta.$$

We now establish the second equations of (12). Using the definition of the covariant derivative and the definition of $\rho_{\alpha\beta}(\boldsymbol{\eta})$, we deduce that

$$\begin{aligned}\lambda_{\alpha|\sigma}&=\partial_\sigma\lambda_\alpha-\Gamma^\nu_{\sigma\alpha}\lambda_\nu=\partial_\sigma(\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_3)-\Gamma^\nu_{\sigma\alpha}(\partial_\nu\boldsymbol{\eta}\cdot\mathbf{a}_3)\\ &=\partial_{\sigma\alpha}\boldsymbol{\eta}\cdot\mathbf{a}_3+\partial_\alpha\boldsymbol{\eta}\cdot\partial_\sigma\mathbf{a}_3-\Gamma^\nu_{\sigma\alpha}(\partial_\nu\boldsymbol{\eta}\cdot\mathbf{a}_3)\\ &=(\partial_{\sigma\alpha}\boldsymbol{\eta}-\Gamma^\nu_{\sigma\alpha}\partial_\nu\boldsymbol{\eta})\cdot\mathbf{a}_3-b^\nu_\sigma\partial_\alpha\boldsymbol{\eta}\cdot\mathbf{a}_\nu\\ &=\rho_{\sigma\alpha}-b^\nu_\sigma(\gamma_{\alpha\nu}+\lambda_{\alpha\nu}),\end{aligned}$$

which constitutes the desired equations.

Finally, we establish the Saint-Venant equations on a surface as a consequence of the Ricci identities

$$\begin{aligned}\lambda_{\alpha\beta|\sigma\tau}-\lambda_{\alpha\beta|\tau\sigma}&=\lambda_{\nu\beta}R^\nu_{\cdot\alpha\sigma\tau}+\lambda_{\alpha\nu}R^\nu_{\cdot\beta\sigma\tau},\\ \lambda_{\alpha|\sigma\tau}-\lambda_{\alpha|\tau\sigma}&=\lambda_\nu R^\nu_{\cdot\alpha\sigma\tau}.\end{aligned}\tag{13}$$

First, using the expressions (12) of $\lambda_{\alpha\beta|\sigma}$ and $\lambda_{\alpha|\sigma}$, we deduce that the second-order covariant derivatives $\lambda_{\alpha\beta|\sigma\tau}$ and $\lambda_{\alpha|\sigma\tau}$, which belong to the space $H^{-2}(\omega)$, satisfy

$$\begin{aligned}\lambda_{\alpha\beta|\sigma\tau}&=\gamma_{\sigma\beta|\alpha\tau}-\gamma_{\sigma\alpha|\beta\tau}-b_{\alpha\sigma}\lambda_{\beta|\tau}+b_{\beta\sigma}\lambda_{\alpha|\tau}-b_{\alpha\sigma|\tau}\lambda_\beta+b_{\beta\sigma|\tau}\lambda_\alpha\\ &=\gamma_{\sigma\beta|\alpha\tau}-\gamma_{\sigma\alpha|\beta\tau}-b_{\alpha\sigma}(\rho_{\tau\beta}-b^\nu_\tau\gamma_{\beta\nu}-b^\nu_\tau\lambda_{\beta\nu})\\ &\qquad+b_{\beta\sigma}(\rho_{\tau\alpha}-b^\nu_\tau\gamma_{\alpha\nu}-b^\nu_\tau\lambda_{\alpha\nu})-b_{\alpha\sigma|\tau}\lambda_\beta+b_{\beta\sigma|\tau}\lambda_\alpha\\ &=\gamma_{\sigma\beta|\alpha\tau}-\gamma_{\sigma\alpha|\beta\tau}+b_{\beta\sigma}\rho_{\tau\alpha}-b_{\alpha\sigma}\rho_{\tau\beta}+b_{\alpha\sigma}b^\nu_\tau\gamma_{\beta\nu}-b_{\beta\sigma}b^\nu_\tau\gamma_{\alpha\nu}\\ &\qquad+b_{\alpha\sigma}b^\nu_\tau\lambda_{\beta\nu}-b_{\beta\sigma}b^\nu_\tau\lambda_{\alpha\nu}-b_{\alpha\sigma|\tau}\lambda_\beta+b_{\beta\sigma|\tau}\lambda_\alpha,\end{aligned}$$

and

$$\begin{aligned}\lambda_{\alpha|\sigma\tau}&=\rho_{\sigma\alpha|\tau}-b^\nu_\sigma\gamma_{\alpha\nu|\tau}-b^\nu_\sigma\lambda_{\alpha\nu|\tau}-b^\nu_{\sigma|\tau}\gamma_{\alpha\nu}-b^\nu_{\sigma|\tau}\lambda_{\alpha\nu}\\ &=\rho_{\sigma\alpha|\tau}-b^\nu_\sigma\gamma_{\alpha\nu|\tau}-b^\nu_\sigma(\gamma_{\tau\nu|\alpha}-\gamma_{\tau\alpha|\nu}-b_{\alpha\tau}\lambda_\nu+b_{\nu\tau}\lambda_\alpha)\\ &\qquad-b^\nu_{\sigma|\tau}\gamma_{\alpha\nu}-b^\nu_{\sigma|\tau}\lambda_{\alpha\nu}.\end{aligned}$$

Using these expressions and the relations

$$b_{\alpha\sigma|\tau}=b_{\alpha\tau|\sigma}\text{ and }b^\nu_{\sigma|\tau}=b^\nu_{\tau|\sigma}\text{ and }b^\nu_\sigma b_{\nu\tau}=b^\nu_\tau b_{\nu\sigma},$$

we next deduce from the Ricci identities (13) that

$$\begin{aligned}
&\gamma_{\sigma\beta|\alpha\tau} - \gamma_{\sigma\alpha|\beta\tau} - \gamma_{\tau\beta|\alpha\sigma} + \gamma_{\tau\alpha|\beta\sigma} + b_{\beta\sigma}\rho_{\tau\alpha} - b_{\alpha\sigma}\rho_{\tau\beta} \\
&\quad - b_{\beta\tau}\rho_{\sigma\alpha} + b_{\alpha\tau}\rho_{\sigma\beta} + (b_{\alpha\sigma}b^\nu_\tau - b_{\alpha\tau}b^\nu_\sigma)\gamma_{\beta\nu} - (b_{\beta\sigma}b^\nu_\tau - b_{\beta\tau}b^\nu_\sigma)\gamma_{\alpha\nu} \\
&\quad + (b_{\alpha\sigma}b^\nu_\tau - b_{\alpha\tau}b^\nu_\sigma)\lambda_{\beta\nu} - (b_{\beta\sigma}b^\nu_\tau - b_{\beta\tau}b^\nu_\sigma)\lambda_{\alpha\nu} = \lambda_{\nu\beta}R^\nu_{\cdot\alpha\sigma\tau} + \lambda_{\alpha\nu}R^\nu_{\cdot\beta\sigma\tau},
\end{aligned}$$

and

$$\begin{aligned}
&\rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} - b^\nu_\sigma(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) \\
&\quad + b^\nu_\tau(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}) + (b^\nu_\sigma b_{\alpha\tau} - b^\nu_\tau b_{\alpha\sigma})\lambda_\nu = \lambda_\nu R^\nu_{\cdot\alpha\sigma\tau}.
\end{aligned}$$

But the terms depending on $\lambda_{\alpha\beta}$ and λ_α appearing in both sides of these last two relations cancel thanks to the Gauss equation

$$R^\nu_{\cdot\alpha\sigma\tau} = b_{\alpha\tau}b^\nu_\sigma - b_{\alpha\sigma}b^\nu_\tau.$$

This shows that the compatibility conditions (11) are satisfied. $\square$

Remarks. (1) Equation (12) shows that the antisymmetric tensor field $(\lambda_{\alpha\beta}(\boldsymbol{\eta}))$ and the vector field $(\lambda_\alpha(\boldsymbol{\eta}))$ are uniquely determined by the linearized change of metric tensor $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$ and the linearized change of curvature tensor $(\rho_{\alpha\beta}(\boldsymbol{\eta}))$, respectively up to an antisymmetric matrix field $(\lambda^0_{\alpha\beta})$ and a vector field (λ^0_α) that are constant in each connected component of ω.

(2) The proof of Theorem 2 shows that the Saint Venant equations on a surface are nothing but the Ricci equations applied to the tensors $(\eta_{\beta|\alpha})$ and $(\eta_{3|\alpha})$. To see this, we note that

$$\eta_{3|\alpha} = \lambda_\alpha \text{ and } \eta_{\beta|\alpha} = \gamma_{\alpha\beta} + \lambda_{\alpha\beta}.$$

These relations, combined with those of (13), show that

$$\begin{aligned}
&\eta_{\beta|\alpha\sigma\tau} - \eta_{\beta|\alpha\tau\sigma} - \gamma_{\alpha\beta|\sigma\tau} + \gamma_{\alpha\beta|\tau\sigma} = (\eta_{\beta|\nu} - \gamma_{\nu\beta})R^\nu_{\cdot\alpha\sigma\tau} + (\eta_{\nu|\alpha} - \gamma_{\alpha\nu})R^\nu_{\cdot\beta\sigma\tau}, \\
&\eta_{3|\alpha\sigma\tau} - \eta_{3|\alpha\tau\sigma} = \eta_{3|\nu}R^\nu_{\cdot\alpha\sigma\tau}.
\end{aligned}$$

But the Ricci identity applied to $(\gamma_{\alpha\beta})$ shows that

$$\gamma_{\alpha\beta|\sigma\tau} - \gamma_{\alpha\beta|\tau\sigma} = \gamma_{\nu\beta}R^\nu_{\cdot\alpha\sigma\tau} + \gamma_{\alpha\nu}R^\nu_{\cdot\beta\sigma\tau}.$$

Hence the Saint Venant equations on a surface hold if and only if

$$\begin{aligned}
&\eta_{\beta|\alpha\sigma\tau} - \eta_{\beta|\alpha\tau\sigma} = \eta_{\beta|\nu}R^\nu_{\cdot\alpha\sigma\tau} + \eta_{\nu|\alpha}R^\nu_{\cdot\beta\sigma\tau}, \\
&\eta_{3|\alpha\sigma\tau} - \eta_{3|\alpha\tau\sigma} = \eta_{3|\nu}R^\nu_{\cdot\alpha\sigma\tau}.
\end{aligned}$$

$\square$

5. Recovery of a vector field from the linearized change of metric and curvature tensors

Let ω be a bounded and open subset of $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion. We refer to Section 3 for the definitions of all the notions used below.

We are now in a position to characterize those symmetric matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ that together satisfy the Saint Venant equations on a surface (cf. (14)).

Theorem 3. *Let ω be a simply-connected domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an immersion. Let there be given two symmetric matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ in the space $L^2(\omega; \mathbb{S}^2)$ that together satisfy:*

$$\begin{aligned}\gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu}& \\ = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha}& \qquad (14)\\ \rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu})&\end{aligned}$$

in the distributional sense.

Then there exists a vector field $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ such that

$$\begin{aligned}\gamma_{\alpha\beta} &= \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}) \quad \text{in } L^2(\omega),\\ \rho_{\alpha\beta} &= (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^{\nu}_{\alpha\beta}\partial_\nu \boldsymbol{\eta}) \cdot \mathbf{a}_3 \quad \text{in } H^{-1}(\omega).\end{aligned} \tag{15}$$

Proof. The proof, which is detailed below, consists in first finding an anti-symmetric matrix field $(\lambda_{\alpha\beta}) \in L^2(\omega; \mathbb{A}^2)$ and a vector field $(\lambda_\alpha) \in L^2(\omega; \mathbb{R}^2)$ that together satisfy the equations

$$\begin{aligned}\lambda_{\alpha\beta|\sigma} + b_{\alpha\sigma}\lambda_\beta - b_{\beta\sigma}\lambda_\alpha &= \gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta},\\ \lambda_{\alpha|\sigma} + b^{\nu}_{\sigma}\lambda_{\alpha\nu} &= \rho_{\sigma\alpha} - b^{\nu}_{\sigma}\gamma_{\alpha\nu}.\end{aligned} \tag{16}$$

in the distributional sense, then finding a vector field $\boldsymbol{\eta} \in H^1(\omega; \mathbb{R}^3)$ that satisfies

$$\partial_\alpha \boldsymbol{\eta} = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^\beta + \lambda_\alpha \mathbf{a}^3. \tag{17}$$

The field $\boldsymbol{\eta}$ is then that announced in the statement of the Theorem.

The proof comprises theee steps.

(i) *We first show that the Saint Venant equations on a surface imply that the system* (16) *has a solution.*

Consider any matrix field

$$\boldsymbol{\lambda} := \lambda_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + \lambda_\alpha \mathbf{a}^\alpha \otimes \mathbf{a}^3 - \lambda_\beta \mathbf{a}^3 \otimes \mathbf{a}^\beta,$$

with coefficients $(\lambda_{\alpha\beta}) \in L^2(\omega; \mathbb{A}^2)$ and $(\lambda_\alpha) \in L^2(\omega; \mathbb{R}^2)$. Then its derivatives are given in $H^{-1}(\omega)$ by the relations

$$\begin{aligned}\partial_\sigma \boldsymbol{\lambda} = \partial_\sigma(\lambda_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) + (\partial_\sigma(\lambda_\alpha \mathbf{a}^\alpha)) \otimes \mathbf{a}^3 + \lambda_\alpha \mathbf{a}^\alpha \otimes (\partial_\sigma \mathbf{a}^3)&\\ - \mathbf{a}^3 \otimes (\partial_\sigma(\lambda_\beta \mathbf{a}^\beta)) - \lambda_\beta(\partial_\sigma \mathbf{a}^3) \otimes \mathbf{a}^\beta.&\end{aligned}$$

By using the definition of the covariant derivatives as given in formulas (7) and (8), these formulas become

$$\begin{aligned}\partial_\sigma \boldsymbol{\lambda} &= \lambda_{\alpha\beta|\sigma}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + b_\sigma^\beta \lambda_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^3 + b_\sigma^\alpha \lambda_{\alpha\beta}\mathbf{a}^3 \otimes \mathbf{a}^\beta \\ &\quad + (\lambda_{\alpha|\sigma}\mathbf{a}^\alpha + b_\sigma^\alpha \lambda_\alpha \mathbf{a}^3) \otimes \mathbf{a}^3 - \mathbf{a}^3 \otimes (\lambda_{\beta|\sigma}\mathbf{a}^\beta + b_\sigma^\beta \lambda_\beta \mathbf{a}^3) \\ &\quad - b_{\sigma\beta}\lambda_\alpha \mathbf{a}^\alpha \otimes \mathbf{a}^\beta + b_{\sigma\alpha}\lambda_\beta \mathbf{a}^\alpha \otimes \mathbf{a}^\beta.\end{aligned}$$

By rearranging the terms and using the antisymmetry of $(\lambda_{\alpha\beta})$, we finally obtain the following expression:

$$\begin{aligned}\partial_\sigma \boldsymbol{\lambda} &= (\lambda_{\alpha\beta|\sigma} - b_{\sigma\beta}\lambda_\alpha + b_{\sigma\alpha}\lambda_\beta)\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &\quad + (\lambda_{\alpha|\sigma} + b_\sigma^\beta \lambda_{\alpha\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^3 - (\lambda_{\beta|\sigma} + b_\sigma^\alpha \lambda_{\beta\alpha})\mathbf{a}^3 \otimes \mathbf{a}^\beta.\end{aligned}$$

By comparing this formula with the system (16), we deduce that the latter has a solution if and only if there exists an antisymmetric matrix field $\boldsymbol{\lambda} := \lambda_{ij}\mathbf{a}^i \otimes \mathbf{a}^j$ (i.e., that satisfies $\lambda_{ij} + \lambda_{ji} = 0$) such that

$$\begin{aligned}\partial_\sigma \boldsymbol{\lambda} &= (\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &\quad + (\rho_{\sigma\alpha} - b_\sigma^\nu \gamma_{\alpha\nu})\mathbf{a}^\alpha \otimes \mathbf{a}^3 - (\rho_{\sigma\beta} - b_\sigma^\nu \gamma_{\beta\nu})\mathbf{a}^3 \otimes \mathbf{a}^\beta.\end{aligned} \tag{18}$$

But Theorem 1 shows that this system has a solution, which necessarily is antisymmetric, if and only if

$$\partial_\tau(\partial_\sigma \boldsymbol{\lambda}) = \partial_\sigma(\partial_\tau \boldsymbol{\lambda}) \text{ in } H^{-2}(\omega).$$

So, it remains to compute these second derivatives. We first infer from (18) that

$$\begin{aligned}\partial_\tau(\partial_\sigma \boldsymbol{\lambda}) &= \partial_\tau((\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta) \\ &\quad + \partial_\tau((\rho_{\sigma\alpha} - b_\sigma^\nu \gamma_{\alpha\nu})\mathbf{a}^\alpha) \otimes \mathbf{a}^3 - \mathbf{a}^3 \otimes \partial_\tau((\rho_{\sigma\beta} - b_\sigma^\nu \gamma_{\beta\nu})\mathbf{a}^\beta) \\ &\quad + (\rho_{\sigma\alpha} - b_\sigma^\nu \gamma_{\alpha\nu})\mathbf{a}^\alpha \otimes \partial_\tau \mathbf{a}^3 - (\rho_{\sigma\beta} - b_\sigma^\nu \gamma_{\beta\nu})\partial_\tau \mathbf{a}^3 \otimes \mathbf{a}^\beta.\end{aligned}$$

Once again using the definition of covariant derivatives as given in formulas (7) and (8), we next obtain that

$$\begin{aligned}\partial_\tau(\partial_\sigma \boldsymbol{\lambda}) &= (\gamma_{\sigma\beta|\alpha\tau} - \gamma_{\sigma\alpha|\beta\tau})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta \\ &\quad + b_\tau^\alpha(\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta})\mathbf{a}^3 \otimes \mathbf{a}^\beta + b_\tau^\beta(\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta})\mathbf{a}^\alpha \otimes \mathbf{a}^3 \\ &\quad + ((\rho_{\sigma\alpha|\tau} - b_\sigma^\nu \gamma_{\alpha\nu|\tau}) + (\Gamma_{\tau\sigma}^\mu \rho_{\mu\alpha} - b_{\sigma|\tau}^\nu \gamma_{\alpha\nu} - \Gamma_{\tau\sigma}^\mu b_\mu^\nu \gamma_{\alpha\nu}))\mathbf{a}^\alpha \otimes \mathbf{a}^3 \\ &\quad + b_\tau^\alpha(\rho_{\sigma\alpha} - b_\sigma^\nu \gamma_{\alpha\nu})\mathbf{a}^3 \otimes \mathbf{a}^3 - b_\tau^\beta(\rho_{\sigma\beta} - b_\sigma^\nu \gamma_{\beta\nu})\mathbf{a}^3 \otimes \mathbf{a}^3 \\ &\quad - ((\rho_{\sigma\beta|\tau} - b_\sigma^\nu \gamma_{\beta\nu|\tau}) + (\Gamma_{\tau\sigma}^\mu \rho_{\mu\beta} - b_{\sigma|\tau}^\nu \gamma_{\beta\nu} - \Gamma_{\tau\sigma}^\mu b_\mu^\nu \gamma_{\beta\nu}))\mathbf{a}^3 \otimes \mathbf{a}^\beta \\ &\quad - b_{\tau\beta}(\rho_{\sigma\alpha} - b_\sigma^\nu \gamma_{\alpha\nu})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta + b_{\tau\alpha}(\rho_{\sigma\beta} - b_\sigma^\nu \gamma_{\beta\nu})\mathbf{a}^\alpha \otimes \mathbf{a}^\beta.\end{aligned}$$

By rearranging the terms and using the antisymmetry of $(\lambda_{\alpha\beta})$, we finally obtain the following expression for the second-order derivatives of $\boldsymbol{\lambda}$:

$$\begin{aligned}\partial_\tau(\partial_\sigma\boldsymbol{\lambda}) &= (\gamma_{\sigma\beta|\alpha\tau} - \gamma_{\sigma\alpha|\beta\tau} - b_{\tau\beta}(\rho_{\sigma\alpha} - b_\sigma^\nu\gamma_{\alpha\nu}) + b_{\tau\alpha}(\rho_{\sigma\beta} - b_\sigma^\nu\gamma_{\beta\nu}))\mathbf{a}^\alpha\otimes\mathbf{a}^\beta\\ &\quad - (\rho_{\sigma\beta|\tau} - b_\sigma^\nu\gamma_{\beta\nu|\tau} - b_\tau^\alpha(\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta}))\mathbf{a}^3\otimes\mathbf{a}^\beta\\ &\quad + (\rho_{\sigma\alpha|\tau} - b_\sigma^\nu\gamma_{\alpha\nu|\tau} + b_\tau^\beta(\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta}))\mathbf{a}^\alpha\otimes\mathbf{a}^3\\ &\quad + (\Gamma_{\tau\sigma}^\mu\rho_{\mu\alpha} - b_{\sigma|\tau}^\nu\gamma_{\alpha\nu} - \Gamma_{\tau\sigma}^\mu b_\mu^\nu\gamma_{\alpha\nu})\mathbf{a}^\alpha\otimes\mathbf{a}^3\\ &\quad - (\Gamma_{\tau\sigma}^\mu\rho_{\mu\beta} - b_{\sigma|\tau}^\nu\gamma_{\beta\nu} - \Gamma_{\tau\sigma}^\mu b_\mu^\nu\gamma_{\beta\nu})\mathbf{a}^3\otimes\mathbf{a}^\beta.\end{aligned}$$

Taking into account the symmetries $\Gamma_{\tau\sigma}^\mu = \Gamma_{\sigma\tau}^\mu$ and $b_\sigma^\nu|_\tau = b_\tau^\nu|_\sigma$, we deduce from the above expression of the second-order derivatives of $\boldsymbol{\lambda}$ that

$$\partial_\tau(\partial_\sigma\boldsymbol{\lambda}) = \partial_\sigma(\partial_\tau\boldsymbol{\lambda})$$

if and only if

$$\begin{aligned}&\gamma_{\sigma\beta|\alpha\tau} - \gamma_{\sigma\alpha|\beta\tau} - b_{\tau\beta}(\rho_{\sigma\alpha} - b_\sigma^\nu\gamma_{\alpha\nu}) + b_{\tau\alpha}(\rho_{\sigma\beta} - b_\sigma^\nu\gamma_{\beta\nu})\\ &\qquad\qquad = \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - b_{\sigma\beta}(\rho_{\tau\alpha} - b_\tau^\nu\gamma_{\alpha\nu}) + b_{\sigma\alpha}(\rho_{\tau\beta} - b_\tau^\nu\gamma_{\beta\nu})\\ &\rho_{\sigma\alpha|\tau} - b_\sigma^\nu\gamma_{\alpha\nu|\tau} + b_\tau^\beta(\gamma_{\sigma\beta|\alpha} - \gamma_{\sigma\alpha|\beta}) = \rho_{\tau\alpha|\sigma} - b_\tau^\nu\gamma_{\alpha\nu|\sigma} + b_\sigma^\beta(\gamma_{\tau\beta|\alpha} - \gamma_{\tau\alpha|\beta}).\end{aligned}$$

But these are exactly the Saint-Vanant equations on a surface, since the Gauss equation (5) precisely states that

$$b_{\tau\beta}b_\sigma^\nu - b_{\sigma\beta}b_\tau^\nu = R^\nu_{\cdot\beta\sigma\tau}.$$

(ii) *We show that the symmetry of the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ imply that there exists a solution $\boldsymbol{\eta}\in H^1(\omega;\mathbb{R}^3)$ to the system* (17).

To this end, we need to prove that

$$\partial_\beta((\gamma_{\alpha\sigma} + \lambda_{\alpha\sigma})\mathbf{a}^\sigma + \lambda_\alpha\mathbf{a}^3) = \partial_\alpha((\gamma_{\beta\sigma} + \lambda_{\beta\sigma})\mathbf{a}^\sigma + \lambda_\beta\mathbf{a}^3)$$

in $H^{-1}(\omega)$. Since

$$\begin{aligned}&\partial_\alpha((\gamma_{\beta\sigma} + \lambda_{\beta\sigma})\mathbf{a}^\sigma + \lambda_\beta\mathbf{a}^3)\\ &= (\partial_\alpha(\gamma_{\beta\sigma} + \lambda_{\beta\sigma}))\mathbf{a}^\sigma + (\partial_\alpha\lambda_\beta)\mathbf{a}^3 + (\gamma_{\beta\sigma} + \lambda_{\beta\sigma})(-\Gamma_{\alpha\mu}^\sigma\mathbf{a}^\mu + b_\alpha^\sigma\mathbf{a}^3) - b_{\alpha\mu}\lambda_\beta\mathbf{a}^\mu\\ &= (\partial_\alpha(\gamma_{\beta\sigma} + \lambda_{\beta\sigma}) - \Gamma_{\alpha\sigma}^\mu(\gamma_{\beta\mu} + \lambda_{\beta\mu}) - b_{\alpha\sigma}\lambda_\beta)\mathbf{a}^\sigma + (\partial_\alpha\lambda_\beta + b_\alpha^\sigma(\gamma_{\beta\sigma} + \lambda_{\beta\sigma}))\mathbf{a}^3\\ &= (\gamma_{\beta\sigma|\alpha} + \lambda_{\beta\sigma|\alpha} + \Gamma_{\alpha\beta}^\mu(\gamma_{\mu\sigma} + \lambda_{\mu\sigma}) - b_{\alpha\sigma}\lambda_\beta)\mathbf{a}^\sigma\\ &\quad + (\lambda_{\beta|\alpha} + \Gamma_{\alpha\beta}^\mu\lambda_\mu + b_\alpha^\sigma(\gamma_{\beta\sigma} + \lambda_{\beta\sigma}))\mathbf{a}^3,\end{aligned}$$

and since $\Gamma_{\alpha\beta}^\mu = \Gamma_{\beta\alpha}^\mu$, it suffices to prove that

$$\begin{aligned}\gamma_{\beta\sigma|\alpha} + \lambda_{\beta\sigma|\alpha} - b_{\alpha\sigma}\lambda_\beta &= \gamma_{\alpha\sigma|\beta} + \lambda_{\alpha\sigma|\beta} - b_{\beta\sigma}\lambda_\alpha,\\ \lambda_{\beta|\alpha} + b_\alpha^\sigma(\gamma_{\beta\sigma} + \lambda_{\beta\sigma}) &= \lambda_{\alpha|\beta} + b_\beta^\sigma(\gamma_{\alpha\sigma} + \lambda_{\alpha\sigma}).\end{aligned}$$

In view of the expressions (16) of the covariant derivatives of $\lambda_{\alpha\beta}$ and λ_α, these equations reduce to

$$\begin{aligned}&\gamma_{\beta\sigma|\alpha} + \gamma_{\alpha\sigma|\beta} - \gamma_{\alpha\beta|\sigma} - b_{\beta\alpha}\lambda_\sigma = \gamma_{\alpha\sigma|\beta} + \gamma_{\beta\sigma|\alpha} - \gamma_{\beta\alpha|\sigma} - b_{\alpha\beta}\lambda_\sigma,\\ &\rho_{\alpha\beta} - b_\alpha^\mu\gamma_{\beta\mu} + b_\alpha^\sigma\gamma_{\beta\sigma} = \rho_{\beta\alpha} - b_\beta^\mu\gamma_{\alpha\mu} + b_\beta^\sigma\gamma_{\alpha\sigma},\end{aligned}$$

hence to

$$\gamma_{\alpha\beta|\sigma} = \gamma_{\beta\alpha|\sigma},$$
$$\rho_{\alpha\beta} = \rho_{\beta\alpha}.$$

But these equations are clearly satisfied, since the matrix fields $(\gamma_{\alpha\beta})$ and $(\rho_{\alpha\beta})$ are symmetric. Hence Poincaré theorem (Theorem 1) shows that there exists a vector field $\boldsymbol{\eta} \in L^2(\omega;\mathbb{R}^3)$, unique up to an additive constant vector field, that satisfies

$$\partial_\alpha \boldsymbol{\eta} = (\gamma_{\alpha\beta} + \lambda_{\alpha\beta})\mathbf{a}^\beta + \lambda_\alpha \mathbf{a}^3.$$

Since the right-hand side of this system belongs to $L^2(\omega;\mathbb{R}^3)$, the field $\boldsymbol{\eta}$ belongs in fact to the space $H^1(\omega;\mathbb{R}^3)$.

(iii) *We finally show that the symmetry of the matrix fields* $(\gamma_{\alpha\beta})$ *and* $(\rho_{\alpha\beta})$*, together with the antisymmetry of the matrix fields* $(\lambda_{\alpha\beta})$*, imply that the vector field* $\boldsymbol{\eta}$ *does satisfy equations* (15).

We first infer from the equation (17) that the functions $\lambda_{\alpha\beta}$ and λ_σ are given in terms of the vector field $\boldsymbol{\eta}$ by

$$\gamma_{\beta\sigma} + \lambda_{\beta\sigma} = \partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_\sigma,$$
$$\lambda_\sigma = \partial_\sigma \boldsymbol{\eta} \cdot \mathbf{a}_3.$$

We then deduce from the first equation and the symmetry of $(\gamma_{\alpha\beta})$ and the antisymmetry of $(\lambda_{\alpha\beta})$ that the functions $\gamma_{\alpha\beta}$ satisfy

$$\gamma_{\alpha\beta} = \frac{1}{2}(\gamma_{\alpha\beta} + \lambda_{\alpha\beta} + \gamma_{\beta\alpha} + \lambda_{\beta\alpha}) = \frac{1}{2}(\partial_\alpha \boldsymbol{\eta} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \boldsymbol{\eta}).$$

We now compute the functions $\rho_{\alpha\beta}$ in terms of the vector field $\boldsymbol{\eta}$. We first infer from the second equation of (16) and from the definition of the covariant derivatives that

$$\begin{aligned}\rho_{\alpha\beta} &= \lambda_{\beta|\alpha} + b_\alpha^\sigma(\lambda_{\beta\sigma} + \gamma_{\beta\sigma}) \\ &= \partial_\alpha(\lambda_\sigma \mathbf{a}^\sigma) \cdot \mathbf{a}_\beta + b_\alpha^\sigma(\lambda_{\beta\sigma} + \gamma_{\beta\sigma}).\end{aligned}$$

Using the above expressions of $(\lambda_{\beta\sigma} + \gamma_{\beta\sigma})$ and λ_σ in this equation, we next deduce that

$$\begin{aligned}\rho_{\alpha\beta} &= \partial_\alpha((\partial_\sigma \boldsymbol{\eta} \cdot \mathbf{a}_3)\mathbf{a}^\sigma) \cdot \mathbf{a}_\beta + b_\alpha^\sigma(\partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_\sigma) \\ &= \partial_\alpha(\partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_3) + (\partial_\sigma \boldsymbol{\eta} \cdot \mathbf{a}_3)(\partial_\alpha \mathbf{a}^\sigma \cdot \mathbf{a}_\beta) + b_\alpha^\sigma(\partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_\sigma).\end{aligned}$$

By using the Gauss equations

$$\partial_\alpha \mathbf{a}^\sigma = -\Gamma^\sigma_{\alpha\mu}\mathbf{a}^\mu + b_\alpha^\sigma \mathbf{a}^3,$$

we finally obtain that

$$\begin{aligned}\rho_{\alpha\beta} &= \partial_{\alpha\beta}\boldsymbol{\eta} \cdot \mathbf{a}_3 - b_\alpha^\mu(\partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_\mu) - \Gamma^\sigma_{\alpha\beta}(\partial_\sigma \boldsymbol{\eta} \cdot \mathbf{a}_3) + b_\alpha^\sigma(\partial_\beta \boldsymbol{\eta} \cdot \mathbf{a}_\sigma) \\ &= (\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma \boldsymbol{\eta}) \cdot \mathbf{a}_3.\end{aligned}$$

$\square$

Remark. The uniqueness result established in Ciarlet & C. Mardare [4, Theorem 3] shows that any vector field $\tilde{\boldsymbol{\eta}} \in H^1(\omega;\mathbb{R}^3)$ that satisfies

$$\gamma_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\beta \tilde{\boldsymbol{\eta}}) \text{ in } L^2(\omega;\mathbb{S}^2)$$

is necessarily of the form

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{\eta}(y) + (\mathbf{a} + \mathbf{b} \wedge \boldsymbol{\theta}(y)) \text{ for almost all } y \in \omega,$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors in $\mathbb{R}^3$. □

6. The linearized Gauss and Codazzi-Mainardi equations

The objective of this Section is to show that the Saint Venant equations on a surface are nothing but an infinitesimal version of Gauss and Codazzi-Mainardi equations. These last equations are recalled in the next theorem, which is a straighforward extension of a well-known result for smooth surfaces in Differential Geometry:

Theorem 4. *Let ω be a domain in $\mathbb{R}^2$, let $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega;\mathbb{R}^3)$ be an immersion, and let the matrix fields $(a_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$ and $(b_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega;\mathbb{S}^2)$, $p>2$, be defined by*

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \text{ and } b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 \text{ in } \omega, \tag{19}$$

where

$$\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$

Then the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ together satisfy the Gauss and Codazzi-Mainardi equations, viz.,

$$\begin{aligned} &R^{\nu}_{\cdot\alpha\sigma\tau} := \partial_\sigma \Gamma^\nu_{\alpha\tau} - \partial_\tau \Gamma^\nu_{\alpha\sigma} + \Gamma^\varphi_{\alpha\tau}\Gamma^\nu_{\varphi\sigma} - \Gamma^\varphi_{\alpha\sigma}\Gamma^\nu_{\varphi\tau} = b_{\alpha\tau}b^\nu_\sigma - b_{\alpha\sigma}b^\nu_\tau, \\ &\partial_\sigma b_{\alpha\tau} - \partial_\tau b_{\alpha\sigma} + \Gamma^\mu_{\alpha\tau} b_{\mu\sigma} - \Gamma^\mu_{\alpha\sigma} b_{\mu\tau} = 0, \end{aligned} \tag{20}$$

in the distributional sense.

Proof. Since $W^{1,p}_{\mathrm{loc}}(\omega) \subset \mathcal{C}^0(\omega)$ by the Sobolev imbedding theorem and since $\det(a_{\alpha\beta}) > 0$ in ω (the matrix $(a_{\alpha\beta}(y))$ being positive definite for all $y \in \omega$ by assumption), the definition of the inverse of a matrix shows that $(a^{\tau\nu}) = (a_{\alpha\beta})^{-1} \in W^{1,p}_{\mathrm{loc}}(\omega;\mathbb{S}^2_>)$. Hence the Christoffel symbols

$$\Gamma^\tau_{\alpha\beta} := \frac{1}{2}a^{\tau\sigma}(\partial_\alpha a_{\beta\sigma} + \partial_\beta a_{\alpha\sigma} - \partial_\sigma a_{\alpha\beta})$$

belong to the space $L^p_{\mathrm{loc}}(\omega)$.

Let the vectors fields $\mathbf{a}^j$ by defined by $\mathbf{a}_i \cdot \mathbf{a}^j = \delta^j_i$ in ω. Then we deduce from the relations (19) that

$$\partial_\sigma a_{\alpha\beta} = \partial_\sigma \mathbf{a}_\alpha \cdot \mathbf{a}_\beta + \mathbf{a}_\alpha \cdot \partial_\sigma \mathbf{a}_\beta.$$

These relations, combined with the definition of the Christoffel symbols and with the relations $\mathbf{a}^\tau = a^{\tau\sigma}\mathbf{a}_\sigma$, imply that

$$\Gamma^\tau_{\alpha\beta} = a^{\tau\sigma}(\mathbf{a}_\sigma \cdot \partial_\alpha \mathbf{a}_\beta) = \mathbf{a}^\tau \cdot \partial_\alpha \mathbf{a}_\beta.$$

Note that relations (19) also imply that

$$b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}^3.$$

Since the vectors $\mathbf{a}_i(y)$ form a basis in $\mathbb{R}^3$ for all $y \in \omega$, we deduce that

$$\partial_\alpha \mathbf{a}_\beta = \Gamma^\tau_{\alpha\beta}\mathbf{a}_\tau + b_{\alpha\beta}\mathbf{a}_3 \text{ in } L^p_{\text{loc}}(\omega;\mathbb{R}^3).$$

Because

$$b^\tau_\alpha = a^{\tau\sigma}b_{\alpha\sigma} = -a^{\tau\sigma}\mathbf{a}_\beta \cdot \partial_\alpha \mathbf{a}^3 = -\mathbf{a}_\tau \cdot \partial_\alpha \mathbf{a}^3,$$

and

$$\partial_\alpha \mathbf{a}_3 \cdot \mathbf{a}^3 = \partial_\alpha \mathbf{a}_3 \cdot \mathbf{a}_3 = \frac{1}{2}\partial_\alpha(\mathbf{a}_3 \cdot \mathbf{a}_3) = 0,$$

we likewise deduce that

$$\partial_\alpha \mathbf{a}_3 = -b^\tau_\alpha \mathbf{a}_\tau \text{ in } L^p_{\text{loc}}(\omega;\mathbb{R}^3).$$

Using now the commutativity of the second-order derivatives of the vector fields $\mathbf{a}_i$ in the sense of distributions, we deduce from the above relations that, for all τ, σ, α,

$$\begin{aligned}
&\partial_\tau(\Gamma^\beta_{\sigma\alpha}\mathbf{a}_\beta + b_{\sigma\alpha}\mathbf{a}_3) = \partial_\sigma(\Gamma^\beta_{\tau\alpha}\mathbf{a}_\beta + b_{\tau\alpha}\mathbf{a}_3),\\
&\partial_\tau(b^\beta_\sigma \mathbf{a}_\beta) = \partial_\sigma(b^\beta_\tau \mathbf{a}_\beta)
\end{aligned}$$

in the distributional sense. Consequently,

$$\begin{aligned}
&\partial_\tau\Gamma^\beta_{\sigma\alpha}\mathbf{a}_\beta + \Gamma^\beta_{\sigma\alpha}(\Gamma^\mu_{\tau\beta}\mathbf{a}_\mu + b_{\tau\beta}\mathbf{a}_3) + \partial_\tau b_{\sigma\alpha}\mathbf{a}_3 - b^\mu_\tau b_{\sigma\alpha}\mathbf{a}_\mu\\
&\qquad = \partial_\sigma\Gamma^\beta_{\tau\alpha}\mathbf{a}_\beta + \Gamma^\beta_{\tau\alpha}(\Gamma^\mu_{\sigma\beta}\mathbf{a}_\mu + b_{\sigma\beta}\mathbf{a}_3) + \partial_\sigma b_{\tau\alpha}\mathbf{a}_3 - b^\mu_\sigma b_{\tau\alpha}\mathbf{a}_\mu,\\
&\partial_\tau b^\beta_\sigma \mathbf{a}_\beta + b^\beta_\sigma(\Gamma^\mu_{\tau\beta}\mathbf{a}_\mu + b_{\tau\beta}\mathbf{a}_3) = \partial_\sigma b^\beta_\tau \mathbf{a}_\beta + b^\beta_\tau(\Gamma^\mu_{\sigma\beta}\mathbf{a}_\mu + b_{\sigma\beta}\mathbf{a}_3)
\end{aligned}$$

which can also be written as

$$\begin{aligned}
&(\partial_\tau\Gamma^\mu_{\sigma\alpha} + \Gamma^\beta_{\sigma\alpha}\Gamma^\mu_{\tau\beta} - b^\mu_\tau b_{\sigma\alpha})\mathbf{a}_\mu + (\partial_\tau b_{\sigma\alpha} + \Gamma^\beta_{\sigma\alpha}b_{\tau\beta})\mathbf{a}_3\\
&\qquad = (\partial_\sigma\Gamma^\mu_{\tau\alpha} + \Gamma^\beta_{\tau\alpha}\Gamma^\mu_{\sigma\beta} - b^\mu_\sigma b_{\tau\alpha})\mathbf{a}_\mu + (\partial_\sigma b_{\tau\alpha} + \Gamma^\beta_{\tau\alpha}b_{\sigma\beta})\mathbf{a}_3\\
&(\partial_\tau b^\mu_\sigma + \Gamma^\mu_{\tau\beta}b^\beta_\sigma)\mathbf{a}_\mu + b^\beta_\sigma b_{\tau\beta}\mathbf{a}_3 = (\partial_\sigma b^\mu_\tau + \Gamma^\mu_{\sigma\beta}b^\beta_\tau)\mathbf{a}_\mu + b^\beta_\tau b_{\sigma\beta}\mathbf{a}_3.
\end{aligned}$$

These equations are satisfied if and only if the equations (20), which are the Gauss and Codazzi-Mainardi equations associated with the two fundamental forms $a_{\alpha\beta}$ and $b_{\alpha\beta}$, vanish in ω. $\square$

Remarkably, the converse of Theorem 4 is also true :

Theorem 5. *Let ω be a connected and simply-connected open subset of $\mathbb{R}^2$ and let $a_{\alpha\beta} \in W^{1,p}_{\text{loc}}(\omega;\mathbb{S}^2_>)$ and $b_{\alpha\beta} \in W^{1,p}_{\text{loc}}(\omega;\mathbb{S}^2)$, $p > 2$, be two matrix fields that satisfy the Gauss and Codazzi-Mainardi equations, namely*

$$\begin{aligned}
&R^\nu_{\cdot\alpha\sigma\tau} := \partial_\sigma\Gamma^\nu_{\alpha\tau} - \partial_\tau\Gamma^\nu_{\alpha\sigma} + \Gamma^\varphi_{\alpha\tau}\Gamma^\nu_{\varphi\sigma} - \Gamma^\varphi_{\alpha\sigma}\Gamma^\nu_{\varphi\tau} = b_{\alpha\tau}b^\nu_\sigma - b_{\alpha\sigma}b^\nu_\tau,\\
&\partial_\sigma b_{\alpha\tau} - \partial_\tau b_{\alpha\gamma} + \Gamma^\mu_{\alpha\tau}b_{\mu\sigma} - \Gamma^\mu_{\alpha\sigma}b_{\mu\sigma} = 0,
\end{aligned}$$

in the distributional sense.

Then there exists an immersion $\boldsymbol{\theta} \in W^{2,p}_{\mathrm{loc}}(\omega; \mathbb{R}^3)$ *such that*

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \text{ and } b_{\alpha\beta} = \partial_\alpha \mathbf{a}_\beta \cdot \mathbf{a}_3 \text{ in } \omega, \tag{21}$$

where

$$\mathbf{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \mathbf{a}_3 := \frac{\mathbf{a}_1 \wedge \mathbf{a}_2}{|\mathbf{a}_1 \wedge \mathbf{a}_2|}.$$

Proof. See the proof of Theorem 9 in S. Mardare [7]. □

Our final objective is to show that Theorems 2 and 3 are in fact "infinitesimal" versions of Theorems 4 and 5, respectively. To this end, we will show that the *Saint-Venant equations on a surface coincide with the linearized Gauss and Codazzi Mainardi equations*:

Theorem 6. *Let* ω *be an open subset of* $\mathbb{R}^2$ *and let* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ *be an immersion. For some* $p > 2$*, let there be given symmetric matrix fields* $(\gamma_{\alpha\beta}) \in W^{1,p}_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ *and* $(\rho_{\alpha\beta}) \in L^p_{\mathrm{loc}}(\omega; \mathbb{S}^2)$ *such that the matrix fields* $(a_{\alpha\beta} + \varepsilon\gamma_{\alpha\beta})$ *and* $(b_{\alpha\beta} + \varepsilon\rho_{\alpha\beta})$ *satisfy the Gauss and Codazzi Mainardi equations for all* $\varepsilon > 0$ *small enough.*

Then the linear part with respect to ε *in the Gauss and Codazzi-Mainardi equations associated with the matrix fields* $(a_{\alpha\beta} + \varepsilon\gamma_{\alpha\beta})$ *and* $(b_{\alpha\beta} + \varepsilon\rho_{\alpha\beta})$ *coincide with the Saint-Venant equations on the surface* $S = \boldsymbol{\theta}(\overline{\omega})$*, i.e.,*

$$\begin{aligned} \gamma_{\sigma\alpha|\beta\tau} + \gamma_{\tau\beta|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\sigma\beta|\alpha\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\alpha\nu} & \\ = b_{\tau\alpha}\rho_{\sigma\beta} + b_{\sigma\beta}\rho_{\tau\alpha} - b_{\sigma\alpha}\rho_{\tau\beta} - b_{\tau\beta}\rho_{\sigma\alpha} & \\ \rho_{\sigma\alpha|\tau} - \rho_{\tau\alpha|\sigma} = b^{\nu}_{\sigma}(\gamma_{\alpha\nu|\tau} + \gamma_{\tau\nu|\alpha} - \gamma_{\tau\alpha|\nu}) - b^{\nu}_{\tau}(\gamma_{\alpha\nu|\sigma} + \gamma_{\sigma\nu|\alpha} - \gamma_{\sigma\alpha|\nu}). & \end{aligned} \tag{22}$$

Proof. It suffices to prove the equality between the linearized Gauss and Codazzi Mainardi equations and the Saint-Venant equations on every compact subset of ω. Hence we may assume in what follows that $(\gamma_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2)$ and $(\rho_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2)$.

For all $\varepsilon > 0$, define the matrix fields

$$\begin{aligned} (a_{\alpha\beta}(\varepsilon)) &:= (a_{\alpha\beta}) + \varepsilon(\gamma_{\alpha\beta}) \in W^{1,p}(\omega; \mathbb{S}^2), \\ (b_{\alpha\beta}(\varepsilon)) &:= (b_{\alpha\beta}) + \varepsilon(\rho_{\alpha\beta}) \in L^p(\omega; \mathbb{S}^2). \end{aligned}$$

Since $W^{1,p}(\omega) \subset \mathcal{C}^0(\overline{\omega})$ by the Sobolev embedding theorem, there exists a number $\varepsilon_0 > 0$ such that, for all $0 \le \varepsilon < \varepsilon_0$, the matrix field $(a_{\alpha\beta}(\varepsilon))$ is positive definite in $\overline{\omega}$. As in the proof Theorem 4, this implies that $a^{\sigma\tau}(\varepsilon) \in W^{1,p}(\omega)$, where $(a^{\sigma\tau}(\varepsilon)) = (a_{\alpha\beta}(\varepsilon))^{-1}$ denotes the inverse of the matrix field $(a_{\alpha\beta}(\varepsilon))$. Hence the Christoffel symbols

$$\begin{aligned} \Gamma_{\alpha\beta\sigma}(\varepsilon) &:= \frac{1}{2}\left\{\partial_\alpha a_{\sigma\beta}(\varepsilon) + \partial_\beta a_{\alpha\sigma}(\varepsilon) - \partial_\sigma a_{\alpha\beta}(\varepsilon)\right\}, \\ \Gamma^{\tau}_{\alpha\beta}(\varepsilon) &:= a^{\tau\sigma}(\varepsilon)\Gamma_{\alpha\beta\sigma}(\varepsilon) \end{aligned}$$

and the mixed components of the second fundamental form, viz.,

$$b^{\tau}_{\alpha}(\varepsilon) = a^{\tau\beta}b_{\alpha\beta}(\varepsilon),$$

all belong to the space $L^p(\omega)$. This property implies that the Gauss and Codazzi-Mainardi equations associated with the two forms $(a_{\alpha\beta}(\varepsilon))$ and $(b_{\alpha\beta}(\varepsilon))$ are well defined in the space of distributions.

Recall that the Gauss equations assert that

$$R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) = b_{\alpha\tau}(\varepsilon)b^{\nu}_{\sigma}(\varepsilon) - b_{\alpha\sigma}(\varepsilon)b^{\nu}_{\tau}(\varepsilon),$$

or equivalently that

$$R_{\beta\alpha\sigma\tau}(\varepsilon) = b_{\alpha\tau}(\varepsilon)b_{\sigma\beta}(\varepsilon) - b_{\alpha\sigma}(\varepsilon)b_{\tau\beta}(\varepsilon), \tag{23}$$

where the functions

$$R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) := \partial_\sigma\Gamma^{\nu}_{\tau\alpha}(\varepsilon) - \partial_\tau\Gamma^{\nu}_{\sigma\alpha}(\varepsilon) + \Gamma^{\mu}_{\tau\alpha}(\varepsilon)\Gamma^{\nu}_{\sigma\mu}(\varepsilon) - \Gamma^{\mu}_{\sigma\alpha}(\varepsilon)\Gamma^{\nu}_{\tau\mu}(\varepsilon)$$

are the mixed components of the Riemann curvature tensor associated with the metric tensor $(a_{\sigma\tau}(\varepsilon))$ and

$$R_{\beta\alpha\sigma\tau}(\varepsilon) = a_{\beta\nu}(\varepsilon)R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon)$$

are the covariant components of the Riemann curvature tensor associated with the same metric tensor. Likewise, recall that the Codazzi-Mainardi equations assert that

$$\partial_\sigma b_{\alpha\tau}(\varepsilon) - \partial_\tau b_{\alpha\sigma}(\varepsilon) + \Gamma^{\mu}_{\alpha\tau}(\varepsilon)b_{\mu\sigma}(\varepsilon) - \Gamma^{\mu}_{\alpha\sigma}(\varepsilon)b_{\mu\tau}(\varepsilon) = 0. \tag{24}$$

Note that the fields $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ also satisfy the Gauss and Codazzi-Mainardi equations in ω, that is,

$$\begin{aligned}&R_{\beta\alpha\sigma\tau} = b_{\alpha\tau}b_{\sigma\beta} - b_{\alpha\sigma}b_{\tau\beta},\\ &\partial_\sigma b_{\alpha\tau} - \partial_\tau b_{\alpha\sigma} + \Gamma^{\mu}_{\alpha\tau}b_{\mu\sigma} - \Gamma^{\mu}_{\alpha\sigma}b_{\mu\tau} = 0,\end{aligned}$$

where

$$R_{\beta\alpha\sigma\tau} := a_{\beta\nu}\left(\partial_\sigma\Gamma^{\nu}_{\tau\alpha} - \partial_\tau\Gamma^{\nu}_{\sigma\alpha} + \Gamma^{\mu}_{\tau\alpha}\Gamma^{\nu}_{\sigma\mu} - \Gamma^{\mu}_{\sigma\alpha}\Gamma^{\nu}_{\tau\mu}\right)$$

are the covariant components of the Riemann curvature tensor associated with the metric $(a_{\alpha\beta})$.

In order to compute the linear part of the Gauss and Codazzi-Mainardi equations associated with the fields $(a_{\alpha\beta}(\varepsilon))$ and $(b_{\alpha\beta}(\varepsilon))$, we proceed by expanding all the above functions as power series in ε. We let $\mathcal{O}(\varepsilon^2)$ denote any function f such that $(\varepsilon^{-2}f)$ is bounded in a space that will be specified in each occurence. We then have

$$a_{\alpha\beta}(\varepsilon) = a_{\alpha\beta} + 2\varepsilon\gamma_{\alpha\beta} + \mathcal{O}(\varepsilon^2) \text{ in } W^{1,p}(\omega),$$

and thus

$$a^{\sigma\tau}(\varepsilon) = a^{\sigma\tau} - 2\varepsilon a^{\sigma\alpha}\gamma_{\alpha\beta}a^{\beta\tau} + \mathcal{O}(\varepsilon^2) \text{ in } W^{1,p}(\omega).$$

Consequently,

$$\begin{aligned}\Gamma_{\alpha\beta\sigma}(\varepsilon) &= \Gamma_{\alpha\beta\sigma} + \frac{1}{2}\{\partial_\alpha(a_{\beta\sigma}(\varepsilon) - a_{\beta\sigma}) + \partial_\beta(a_{\alpha\sigma}(\varepsilon) - a_{\alpha\sigma}) - \partial_\sigma(a_{\alpha\beta}(\varepsilon) - a_{\alpha\beta})\}\\ &= \Gamma_{\alpha\beta\sigma} + \varepsilon(\partial_\alpha\gamma_{\beta\sigma} + \partial_\beta\gamma_{\alpha\sigma} - \partial_\sigma\gamma_{\alpha\beta}) + \mathcal{O}(\varepsilon^2) \text{ in } L^p(\omega),\end{aligned}$$

and

$$\begin{aligned}
\Gamma^\tau_{\alpha\beta}(\varepsilon) &= (a^{\tau\sigma} - 2\varepsilon a^{\tau\varphi}\gamma_{\varphi\psi}a^{\psi\sigma} + \mathcal{O}(\varepsilon^2))(\Gamma_{\alpha\beta\sigma} + \varepsilon(\partial_\alpha\gamma_{\sigma\beta} + \partial_\beta\gamma_{\sigma\alpha} - \partial_\sigma\gamma_{\alpha\beta}) + \mathcal{O}(\varepsilon^2))\\
&= a^{\tau\sigma}\Gamma_{\alpha\beta\sigma} + \varepsilon a^{\tau\sigma}(\partial_\alpha\gamma_{\sigma\beta} + \partial_\beta\gamma_{\sigma\alpha} - \partial_\sigma\gamma_{\alpha\beta}) - 2\varepsilon a^{\tau\varphi}\gamma_{\varphi\psi}a^{\psi\sigma}\Gamma_{\alpha\beta\sigma} + \mathcal{O}(\varepsilon^2)\\
&= \Gamma^\tau_{\alpha\beta} + \varepsilon a^{\tau\sigma}(\partial_\alpha\gamma_{\sigma\beta} + \partial_\beta\gamma_{\sigma\alpha} - \partial_\sigma\gamma_{\alpha\beta} - 2\Gamma^\mu_{\alpha\beta}\gamma_{\sigma\mu}) + \mathcal{O}(\varepsilon^2)\\
&= \Gamma^\tau_{\alpha\beta} + \varepsilon a^{\tau\sigma}(\gamma_{\sigma\beta|\alpha} + \gamma_{\sigma\alpha|\beta} - \gamma_{\alpha\beta|\sigma}) + \mathcal{O}(\varepsilon^2) \text{ in } L^p(\omega).
\end{aligned}$$

Defining the functions

$$H_{\alpha\beta\sigma} := \gamma_{\sigma\beta|\alpha} + \gamma_{\sigma\alpha|\beta} - \gamma_{\alpha\beta|\sigma} \text{ and } H^\tau_{\alpha\beta} := a^{\tau\sigma}H_{\alpha\beta\sigma},$$

we thus obtain the following relations in $L^p(\omega)$:

$$\begin{aligned}
\Gamma^\tau_{\alpha\beta}(\varepsilon) &= \Gamma^\tau_{\alpha\beta} + \varepsilon H^\tau_{\alpha\beta} + \mathcal{O}(\varepsilon^2)\\
\Gamma_{\alpha\beta\sigma}(\varepsilon) &= \Gamma_{\alpha\beta\sigma} + \varepsilon H_{\alpha\beta\sigma} + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Using these relations in the definition of $R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon)$, we next deduce that

$$\begin{aligned}
R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) = R^{\nu}_{\cdot\alpha\sigma\tau} + \varepsilon(&\partial_\sigma H^\nu_{\tau\alpha} - \partial_\tau H^\nu_{\sigma\alpha} + \Gamma^\mu_{\tau\alpha}H^\nu_{\sigma\mu} + H^\mu_{\tau\alpha}\Gamma^\nu_{\sigma\mu}\\
&- \Gamma^\mu_{\sigma\alpha}H^\nu_{\tau\mu} - H^\mu_{\sigma\alpha}\Gamma^\nu_{\tau\mu}) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

in the space $W^{-1,p}(\omega)$, hence also in the space $H^{-1}(\omega)$.

Let the covariant derivatives of the tensor fields $(H^\nu_{\tau\alpha})$, of $(H_{\nu\tau\alpha})$, and of $(a^{\tau\nu})$ be denoted by

$$\begin{aligned}
H^\nu_{\tau\alpha}|_\sigma &:= \partial_\sigma H^\nu_{\tau\alpha} - \Gamma^\mu_{\sigma\tau}H^\nu_{\mu\alpha} - \Gamma^\mu_{\sigma\alpha}H^\nu_{\tau\mu} + \Gamma^\nu_{\sigma\mu}H^\mu_{\tau\alpha},\\
H_{\tau\alpha\nu|\sigma} &:= \partial_\sigma H_{\tau\alpha\nu} - \Gamma^\mu_{\sigma\tau}H_{\mu\alpha\nu} - \Gamma^\mu_{\sigma\alpha}H_{\tau\mu\nu} - \Gamma^\mu_{\sigma\nu}H_{\tau\alpha\mu},\\
a^{\tau\nu}|_\sigma &:= \partial_\sigma a^{\tau\nu} + \Gamma^\tau_{\sigma\mu}a^{\mu\nu} + \Gamma^\nu_{\sigma\mu}a^{\tau\mu}.
\end{aligned}$$

Then they satisfy the following relation

$$H^\nu_{\tau\alpha}|_\sigma = a^{\nu\mu}H_{\tau\alpha\mu|\sigma} + a^{\nu\mu}|_\sigma H_{\tau\alpha\mu} \text{ in } H^{-1}(\omega).$$

Moreover, the definition of the Christoffel symbols associated with the metric tensor $(a_{\sigma\tau})$ shows that

$$\begin{aligned}
a^{\tau\nu}|_\sigma &= \partial_\sigma a^{\tau\nu} + \frac{1}{2}(a^{\tau\varphi}a^{\psi\nu} + a^{\tau\psi}a^{\varphi\nu})\Gamma_{\sigma\psi\varphi}\\
&= \partial_\sigma a^{\tau\nu} + \frac{1}{2}\{a^{\tau\varphi}(a^{\psi\nu}\partial_\sigma a_{\varphi\psi}) + a^{\tau\psi}(a^{\varphi\nu}\partial_\sigma a_{\varphi\psi})\\
&\qquad + (a^{\tau\varphi}a^{\psi\nu} + a^{\tau\psi}a^{\varphi\nu})(\partial_\psi a_{\sigma\varphi} - \partial_\varphi a_{\sigma\psi})\}\\
&= \partial_\sigma a^{\tau\nu} - \frac{1}{2}\{a^{\tau\varphi}(\partial_\sigma a^{\psi\nu})a_{\varphi\psi} + a^{\tau\psi}(\partial_\sigma a^{\varphi\nu})\partial_\sigma a_{\varphi\psi}\}\\
&= 0 \text{ in } \overline{\omega},
\end{aligned}$$

which, combined with the previous relation, implies that

$$H^\nu_{\tau\alpha}|_\sigma = a^{\nu\mu}H_{\tau\alpha\mu|\sigma} \text{ in } H^{-1}(\omega).$$

Using this relation in the expression of $R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon)$ yields the following relations in $H^{-1}(\omega)$:

$$\begin{aligned} R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) - R^{\nu}_{\cdot\alpha\sigma\tau} &= \varepsilon(H^{\nu}_{\tau\alpha}|_{\sigma} - H^{\nu}_{\sigma\alpha}|_{\tau}) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon a^{\nu\mu}(H_{\tau\alpha\mu|\sigma} - H_{\sigma\alpha\mu|\tau}) + \mathcal{O}(\varepsilon^2) \\ &= \varepsilon a^{\nu\mu}(\gamma_{\mu\tau|\alpha\sigma} + \gamma_{\mu\alpha|\tau\sigma} - \gamma_{\tau\alpha|\mu\sigma} - \gamma_{\mu\sigma|\alpha\tau} - \gamma_{\mu\alpha|\sigma\tau} + \gamma_{\sigma\alpha|\mu\tau}) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

and

$$\begin{aligned} R_{\beta\alpha\sigma\tau}(\varepsilon) - R_{\beta\alpha\sigma\tau} &= a_{\beta\nu}(\varepsilon)R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) - a_{\beta\nu}R^{\nu}_{\cdot\alpha\sigma\tau} \\ &= a_{\beta\nu}(R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) - R^{\nu}_{\cdot\alpha\sigma\tau}) + (a_{\beta\nu}(\varepsilon) - a_{\beta\nu})R^{\nu}_{\cdot\alpha\sigma\tau}(\varepsilon) \\ &= \varepsilon a_{\beta\nu}a^{\nu\mu}(\gamma_{\mu\tau|\alpha\sigma} + \gamma_{\mu\alpha|\tau\sigma} - \gamma_{\tau\alpha|\mu\sigma} - \gamma_{\mu\sigma|\alpha\tau} - \gamma_{\mu\alpha|\sigma\tau} + \gamma_{\sigma\alpha|\mu\tau}) \\ &\quad + 2\varepsilon\gamma_{\beta\nu}R^{\nu}_{\cdot\alpha\sigma\tau} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Combining these relations with the Ricci identity

$$\gamma_{\mu\alpha|\tau\sigma} - \gamma_{\mu\alpha|\sigma\tau} = R^{\nu}_{\cdot\mu\sigma\tau}\gamma_{\nu\alpha} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\mu\nu},$$

we deduce that, on the one hand,

$$\begin{aligned} &R_{\beta\alpha\sigma\tau}(\varepsilon) - R_{\beta\alpha\sigma\tau} \\ &\quad = \varepsilon(\gamma_{\beta\tau|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\beta\sigma|\alpha\tau} + \gamma_{\sigma\alpha|\beta\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\nu\alpha}) + \mathcal{O}(\varepsilon^2) \end{aligned}$$

in the space $H^{-1}(\omega)$. On the other hand, we have

$$\begin{aligned} b_{\alpha\tau}(\varepsilon)b_{\sigma\beta}(\varepsilon) - b_{\alpha\sigma}(\varepsilon)b_{\tau\beta}(\varepsilon) &= b_{\alpha\tau}b_{\sigma\beta} - b_{\alpha\sigma}b_{\tau\beta} \\ &\quad + \varepsilon(b_{\alpha\tau}\rho_{\sigma\beta} + \rho_{\alpha\tau}b_{\sigma\beta} - b_{\alpha\sigma}\rho_{\tau\beta} - \rho_{\alpha\sigma}b_{\tau\beta}) + \mathcal{O}(\varepsilon^2) \text{ in } L^p(\omega). \end{aligned}$$

Therefore the linear part of the Gauss equations (23) are the following equations in $H^{-1}(\omega)$:

$$\begin{aligned} \gamma_{\beta\tau|\alpha\sigma} - \gamma_{\tau\alpha|\beta\sigma} - \gamma_{\beta\sigma|\alpha\tau} + \gamma_{\sigma\alpha|\beta\tau} + R^{\nu}_{\cdot\alpha\sigma\tau}\gamma_{\beta\nu} - R^{\nu}_{\cdot\beta\sigma\tau}\gamma_{\nu\alpha} \\ = b_{\alpha\tau}\rho_{\sigma\beta} + \rho_{\alpha\tau}b_{\sigma\beta} - b_{\alpha\sigma}\rho_{\tau\beta} - \rho_{\alpha\sigma}b_{\tau\beta}, \end{aligned}$$

which are exactly the first Saint-Venant equations on a surface (see (22)).

We now compute the linear part of the Codazzi-Mainardi equations (24). Using the power series expansions of $b_{\alpha\beta}(\varepsilon)$ and of $\Gamma^{\mu}_{\alpha\beta}(\varepsilon)$, we first deduce that this linear part is given by the following equations in $H^{-1}(\omega)$:

$$\partial_\sigma\rho_{\alpha\tau} - \partial_\tau\rho_{\alpha\sigma} + \Gamma^{\mu}_{\alpha\tau}\rho_{\mu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\rho_{\mu\tau} + H^{\mu}_{\alpha\tau}b_{\mu\sigma} - H^{\mu}_{\alpha\sigma}b_{\mu\tau} = 0.$$

Then, by definition of the covariant derivatives and by definition of the functions $H^{\tau}_{\alpha\beta}$, these last equations are equivalent with

$$\rho_{\alpha\tau|\sigma} - \rho_{\sigma\alpha|\tau} + (\gamma_{\nu\alpha|\sigma} + \gamma_{\nu\sigma|\alpha} + \gamma_{\sigma\alpha|\nu})b^{\nu}_{\alpha} - (\gamma_{\nu\alpha|\tau} + \gamma_{\nu\tau|\alpha} + \gamma_{\tau\alpha|\nu})b^{\nu}_{\sigma} = 0,$$

which are exactly the second Saint-Venant equations on a surface (see (22)).

This completes the proof. $\square$

Remark. In Theorem 6, the field $(\gamma_{\alpha\beta})$ belongs to the space $W^{1,p}_{loc}(\omega;\mathbb{S}^2)$, so as to guarantee that $(a_{\alpha\beta}(\varepsilon)) \in W^{1,p}_{loc}(\omega;\mathbb{S}^2)$, which is the minimal regularity assumption under which the Riemannian curvature tensor $R_{\beta\alpha\sigma\tau}(\varepsilon)$ is well defined in the space of distributions. However, the Saint-Venant equations (22) can be extended by continuity to matrix fields $(\gamma_{\alpha\beta})$ that belong only to the space $L^2_{loc}(\omega; \mathbb{S}^2)$. $\square$

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